Dynamic Completeness and Market Frictions*

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Abstract

This article proposes the first characterization of dynamic completeness in markets with frictions. In frictionless markets with no available arbitrage opportunity, the fundamental theorem of asset pricing states that dynamic completeness is equivalent to having a unique normalized vector of strictly positive event prices under which every investment makes zero profit. First, we show that it is also equivalent to the weaker condition that a supporting event price vector with a zero first-period price does not exist. Then, we demonstrate that there is no arbitrage opportunity in multi-period security markets with bid–ask spreads if, and only if, frictionless no-arbitrage markets support them. Eventually, we prove that the absence of a supporting event price vector with a zero first-period price also characterizes dynamic completeness in markets with bid–ask spreads. On the other hand, we show that having a unique normalized vector of strictly positive event prices supporting these markets is unnecessary for dynamic completeness.

Keywords— security markets microstructure, complete markets, incomplete markets, efficiency, financial innovation, risk sharing, friction, bid–ask spread, present value, market spanning

1 Introduction

This article proposes the first characterization of dynamic completeness in the presence of market frictions. We demonstrate that markets with bid–ask spreads are dynamically complete if, and only if, every frictionless supporting markets have a non-zero event-0 price.

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Dynamic completeness is an eminently desirable property of financial security markets that requires that every contract or security be traded (possibly by replicating them). It ensures that market participants perfectly transfer risk and smooth their consumption intertemporally. Over the past century, financial markets have produced a multitude of innovative products, including many new forms of derivatives, alternative risk transfer products, exchange-traded funds, and variants of tax-deductible equity, to increase risk-sharing opportunities (Van Horne (1985)) and move financial markets towards dynamic completeness (see Allen and Gale (1994) and Tufano (2003)). For example, option contracts reduce significantly the number of securities necessary for dynamic completeness (see Ross (1976), Breeden and Litzenberger (1978), Friesen (1979), Green and Jarrow (1987), Nachman (1988), Baptista (2003)). The remaining sources of dynamic incompleteness are explained by informational problems such as moral hazard, adverse selection, unforecastable events, or the existence of too many events (see Laffont (1989)), short-selling restrictions (see Raab and Schwager (1993)), transaction costs (see Merton (1989) and Ross (1989)), taxes or fees. However, these frictions do not necessarily result in markets incompleteness. For example, Raab and Schwager (1993) provides a sufficient condition for market completeness in $2-$period security markets with short-selling restrictions.

This paper demonstrates that the principal transaction cost when trading stocks, futures contracts, options, or currency pairs (see Kumar (2004)), the bid–ask spread, does not necessarily result in dynamically incomplete markets. Actually, in some cases, suppressing bid–ask spreads makes the markets dynamically incomplete (see example 2.1). Bid–ask spreads represent the remuneration of market makers, key participants of security markets who provide bid and ask offers for securities resulting in a significant improvement of markets' liquidity. The size of bid–ask spreads has been explained by the extent of the competition between market makers (see Tinic and West (1972) Stoll (1978), Ho and Stoll (1983) and Biais, Martimort and Rochet (2000)), by inventory consideration (see Tinic (1972), Garman (1976), Amihud and Mendelson (1980) and Ho and Stoll (1981)), by adverse selection arising from asymmetric information (see Bagehot (1971), Copeland and Galai (1983), Glosten and Milgrom (1985), Kyle (1985) and Glosten (1989)), by the ability of market makers and investors to find counterparties (see Demsetz (1968) and Duffie, Gârleanu and Pedersen (2005)), by the distribution of securities holdings (see Lagos and Rocheteau (2009)), and by the extent of the deployment of algorithmic trading (see Hendershott, Jones and Menkveld (2011)). Additionally, Cohen, Maier, Schwartz and Whitcomb (1981) and Martins-da-Rocha and Vailakis (2010) prove the existence of bid–ask spreads at equilibrium in financial security markets models.

The characterization of dynamic completeness is well-known in frictionless markets (see Magill and Quinzii (1996) or LeRoy and Werner (2014)). In a standard frictionless economy with no arbitrage opportunity available, dynamic completeness is equivalent to having a unique normalized vector of strictly positive event prices under which every investment makes a zero profit. First, we notice that a payoff stream can be generated equivalently by a portfolio strategy (which records the holding in each security at the end of each period) or by a trading strategy (which records the orders passed in each event). However, trading strategies outperform portfolio strategies in the analysis of financial markets in the presence of frictions because they permit the use of more straightforward mathematical methods involving positive spanning. We demonstrate that the set of payoff streams that a trading strategy can generate is equal to the positive span of the payoff matrix. Therefore, we propose a new characterization of the absence of arbitrage opportunity using trading strategies in multi-period security markets with bid-ask spreads. We show that no-arbitrage is equivalent to the existence of supporting frictionless markets with no-arbitrage opportunity. Then, we demonstrate the equivalence between the uniqueness of the vector of strictly positive event prices supporting the market and the weaker condition that a supporting event price vector with a zero first-period price does not exist, in frictionless markets with no arbitrage opportunity. Finally, we show that the uniqueness of the vector of event prices supporting the economy is not necessary for dynamic completeness in the presence of friction. On the other hand, we prove that the absence of supporting event prices vector with zero initial event price remains equivalent to dynamic completeness in markets with bid-ask spreads.

This paper is organized as follows. For the sake of exposition, we first present our results in a particular case of a $3-$period security market with bid–ask spread in Section 2. We introduce the concept of trading strategies and present our characterization of dynamic completeness in this setting. We also determine the minimal number of traded securities necessary for dynamic completeness. Our results are illustrated graphically both in section 2.6 and 2.7. In section 2.8, we provide examples of security markets that are dynamically complete with bid–ask spreads. In Section 3, we extend the results of Section 2 to the general case of multi-period security markets with bid–ask spreads. We provide examples of applications of the results in the conclusion.

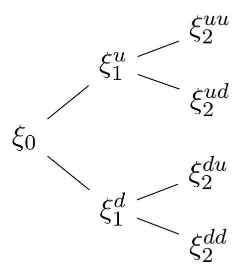
2 Dynamic completeness in 3-period markets with bid–ask spreads

This section presents the characterization of dynamic completeness in a particular case of 3-period security markets with bid–ask spreads. We do not present first the results in a $2-$period security markets as is usually the case because the presence of bid–ask spread has no influence on completeness in these markets (see Remark 2.1) while it is no longer the case in multi-period security markets as the prices at which trades take place impacts future payoffs.

Throughout the section, we assume there is no restriction to short selling and no limitation to the quantity of security purchased and sold at the initial period. We additionally assume agents can infinitely split their orders, and they share the same information structure. Hence incompleteness cannot result from the presence of one of these frictions. We present the general multiperiod case in Section 3.

2.1 The Information Structure

We represent uncertainty about the future by a set of events that can happen at each period. We denote ξ_t an event happening at period t. We denote it ξ when the precision is unnecessary. At period 0, agents do not know which events will realize in the future. At period 1, they know that only a subset of events may happen at period 2. We represent the unfolding of events in the following event tree.

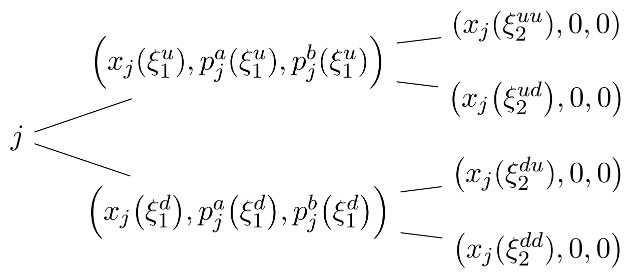


It should be interpreted in the following manner, if ξ_1^u realizes then only events ξ_2^{uu} and ξ^{ud} can happen at period 2. We denote Ξ_0 the set containing only the event ξ_0, Ξ_1 the set of events $\{\xi_1^u, \xi_1^d\}$ and Ξ_2 the set of events $\{\xi_2^{uu}, \xi_2^{ud}, \xi_2^{du}, \xi_2^{dd}\}$.

2.2 The Market Structure

We consider markets in which J securities are traded at period 0 and period 1. To each security j, corresponds a dividend stream represented by a vector $x_j \in \mathbb{R}^6$. We denote $x_j(\xi)$ the dividend paid by security j in event $\xi \in \Xi_1 \cup \Xi_2$ and $x(\xi) \in \mathbb{R}^J$ the dividends paid by each securities in event $\xi \in \Xi_1 \cup \Xi_2$.

There are two types of actors participating in the markets: investors and market makers. Market makers actively quote two-sided markets in a particular security, providing bids and asks. The bid–ask spread, the difference between the buy price and the selling price they propose, compensate them for their services, and the risk they bear for holding securities during several periods. Investors purchase and sell securities to a market maker. They buy at the ask price and sell at the bid price (a market maker buys at the bid price and sells at the ask price). We adopt their perspective in the following. The ask price of security j, denoted $p_j^a(\xi)$, represents the amount spent by an investor to purchase this security in the event $\xi \in \Xi_1 \cup \Xi_2$. Its bid price, $p_j^b(\xi)$, represents the amount received by the investor when he sells security j in event $\xi \in \Xi_1 \cup \Xi_2$. We have period 2 prices for practicality. We set them equal to 0, that is $p_j^a(\xi_2) = p_j^b(\xi_2) = 0$ for all $\xi_2 \in \Xi_2$ and all security j. The spread between the ask and the bid price is the bid–ask spread. We denote $p^a(\xi) \in \mathbb{R}^J$ the vector of securities ask prices and $p^b(\xi) \in \mathbb{R}^J$ the vector of securities bid prices in an event $\xi \in \Xi_1 \cup \Xi_2$. We present the unfolding of the dividends and prices of x_j in a tree by associating a triplet $(x_j(\xi), p_j^a(\xi), p_j^b(\xi))$ to each non-initial node $\xi \in \Xi_1 \cup \Xi_2$.



At each period, agents constitute a portfolio of securities. We denote $h(\xi) \in \mathbb{R}^J$ the portfolio held in an event $\xi \in \Xi_0 \cup \Xi_1 \cup \Xi_2$. The coordinates of $h(\xi)$ can be either positive, negative, or zero. Positivity of the j^{th} coordinate of $h(\xi)$ means that the agent owns security j. Negativity means that she has sold j and owes its dividend to its owner. The triplet $h = (h_0, h_1, h_2)$ is a trading strategy where h_t, $t = 0, 1, 2$, is a vector taking coordinates $h(\xi_t)$ for all $\xi_t \in \Xi_t$.

2.3 Dynamic Completeness and the Set of Available Payoffs

A trading strategy's payoff in a particular event represents the net amount received by the agent after trading in the markets at this period. She first receives the dividends of the portfolio she had constituted at the previous period. Next, she trades on the markets. These two components enter the payoff she receives. Formally, the payoff in an event $\xi_1 \in \Xi_1$ of a trading strategy h is denoted $z(h, p^a, p^b)(\xi_1)$. It is equal to

$$\overbrace{x(\xi_1)h(\xi_0)}^{\text{dividends}} - \overbrace{p^b(\xi_1)\min(h(\xi_1) - h(\xi_0)), 0)}^{\text{sales revenue}} - \overbrace{p^a(\xi_1)\max(h(\xi_1) - h(\xi_0), 0)}^{\text{purchases cost}} \tag{1}$$

where for $(x, y, z) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k$, $z = \max(x, y)$ means $z_i = \max(x_i, y_i)$ for every $i = 1, \dots, k$ and $z = \min(x, y)$ means $z_i = \min(x_i, y_i)$ for every $i = 1, \dots, k$. Since there is no market opened at period 2, the payoff of a trading strategy in an event $\xi_2 \in \Xi_2$ represents solely the difference between the dividends received and due. Formally, the payoff in an event $\xi_2 \in \Xi_2$ of a trading strategy h is denoted $z(h, p^a, p^b)(\xi_2)$. It is equal to $x(\xi_2)h(\xi_2^-)$ where ξ_2^- denotes the immediate predecessor of event ξ_2.

The set of payoff streams that are replicated by portfolio strategies is the set $\mathcal{M}(p)$ equal to

$$\left\{ z \in \mathbb{R}^6 \mid \exists h \text{ s.t. } z(\xi_t) = z(h, p^a, p^b)(\xi_t) \text{ for all } t = 0, 1, 2 \right\}.$$

Markets are dynamically complete if every payoff stream can be replicated. Formally, markets are dynamically complete if $\mathcal{M}(p) = \mathbb{R}^6$. In frictionless markets, we call this set the asset span because it is equal to the span (in the mathematical sense) of a set of payoff streams. In our context, this set is not a span due to bid–ask spreads. Therefore, we call it the set of available payoff streams. In the following section, we provide an equivalent definition for the set of available payoff streams.

2.4 Trading Strategies

First, we introduce the concept of trading strategies. A trading strategy records the unfolding of market orders placed in each event. We denote $b^a(\xi) \in \mathbb{R}_+^J$ the ask orders placed in event $\xi \in \Xi_0 \cup \Xi_1$ and $b^b(\xi) \in \mathbb{R}_+^J$ the bid orders placed in event $\xi \in \Xi_0 \cup \Xi_1$. We emphasize the fact that orders exclusively admit non-negative[1] value as opposed to portfolios that equally admit negative values. Since a portfolio is equal to the sum of ask and bid orders placed at the previous periods, we can recover the orders placed on the market from a portfolios strategy and vice-versa (see also Proposition 2.1). We notice that to a given portfolio strategy h, we can associate the orders placed in the markets in the following manner

$$b^a(\xi_0) = \overbrace{\max(h(\xi_0), 0)}^{\text{ask orders placed at } t=0}, \quad b^b(\xi_0) = \overbrace{-\min(h(\xi_0), 0)}^{\text{bid orders placed at } t=0}$$

and

$$b^a(\xi_1) = \overbrace{\max(h(\xi_1) - h(\xi_0), 0)}^{\text{ask orders placed in event } \xi_1}, \quad b^b(\xi_1) = \overbrace{-\min(h(\xi_1) - h(\xi_0), 0)}^{\text{bid orders placed in event } \xi_1} \text{ for all } \xi_1 \in \Xi_1.$$

At period 0, an investor can place $2J$ different types of orders (the factor 2 stands for the ask and bid orders). At period 1, an investor can place $4J$ different order types (since there are 2 events). Therefore, an investor can place a total of $6J$ different market orders, and a trading strategy is a vector b of $\mathbb{R}_+^{6J}$ equal to

$$\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \\ b^a(\xi_1^u) \\ b^b(\xi_1^u) \\ b^a(\xi_1^d) \\ b^b(\xi_1^d) \end{pmatrix}.$$

[1]We implement the following convention: positive means strictly superior to zero, non-negative means superior or equal to 0, non-positive means inferior or equal to 0 and negative means strictly inferior to 0.

Each market order endows its issuer with a particular payment stream between periods 0 and 2. The payment stream of an ask order placed in event ξ_t, $0 \leqslant t \leqslant T-1$ on security j corresponds to the payment carried out in event ξ_t by the buyer to purchase the security, the dividends he receives in the successor events and 0 otherwise. Similarly, the payment stream of a bid order placed ξ_t, $0 \leqslant t \leqslant T-1$ on j is equal to the payment received by the seller of j in this event, the dividends paid to the buyer in the successor events, and zero otherwise. For example, the payment stream of a buy order placed on security j in event ξ_0 is a vector $\hat{\phi}^a_j(\xi_0)$ equal to

$$\begin{pmatrix} -p^a_j(\xi_0) \\ x_j(\xi_1^u) \\ x_j(\xi_1^d) \\ x_j(\xi_2^{uu}) \\ x_j(\xi_2^{ud}) \\ x_j(\xi_2^{du}) \\ x_j(\xi_2^{dd}) \end{pmatrix} \begin{array}{l} \text{payment in event } \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{array}.$$

Similarly, the payment stream of a sell order placed on security j in event ξ_1^u is a vector $\hat{\phi}^b_j(\xi_1^u)$ equal to

$$\begin{pmatrix} 0 \\ p^b_j(\xi_1^u) \\ 0 \\ -x_j(\xi_2^{uu}) \\ -x_j(\xi_2^{ud}) \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \text{payment in event } \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{array}.$$

We regroup the payment streams of bid and ask orders issued in event ξ_0 in a $7 \times 2J$ matrix $\phi(\xi_0)$ equal to

$$\begin{array}{c} \\ \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{\phi}^a_1(\xi_0) & \dots & \hat{\phi}^a_J(\xi_0) & \hat{\phi}^b_1(\xi_0) & \dots & \hat{\phi}^b_J(\xi_0) \end{array} \\ \begin{pmatrix} -p^a_1(\xi_0) & \dots & -p^a_J(\xi_0) & p^b_1(\xi_0) & \dots & p^b_J(\xi_0) \\ x_1(\xi_1^u) & \dots & x_J(\xi_1^u) & -x_1(\xi_1^u) & \dots & -x_J(\xi_1^u) \\ x_1(\xi_1^d) & \dots & x_J(\xi_1^d) & -x_1(\xi_1^d) & \dots & -x_J(\xi_1^d) \\ x_1(\xi_2^{uu}) & \dots & x_J(\xi_2^{uu}) & -x_1(\xi_2^{uu}) & \dots & -x_J(\xi_2^{uu}) \\ x_1(\xi_2^{ud}) & \dots & x_J(\xi_2^{ud}) & -x_1(\xi_2^{ud}) & \dots & -x_J(\xi_2^{ud}) \\ x_1(\xi_2^{du}) & \dots & x_J(\xi_2^{du}) & -x_1(\xi_2^{du}) & \dots & -x_J(\xi_2^{du}) \\ x_1(\xi_2^{dd}) & \dots & x_J(\xi_2^{dd}) & -x_1(\xi_2^{dd}) & \dots & -x_J(\xi_2^{dd}) \end{pmatrix} \end{array}.$$

Similarly, we regroup the payment streams of bid and ask orders issued in event $\xi_1^u \in \Xi_1$

in a $7 \times 2J$ matrix $\hat{\phi}(\xi_1^u)$ equal to

$$
\begin{array}{c}
 \\ \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\hat{\phi}_1^a(\xi_1^u) & \dots & \hat{\phi}_J^a(\xi_1^u) & \hat{\phi}_1^b(\xi_1^u) & \dots & \hat{\phi}_J^b(\xi_1^u)
\end{array} \\
\begin{pmatrix}
0 & \dots & 0 & 0 & \dots & 0 \\
-p_1^a(\xi_1^u) & \dots & -p_J^a(\xi_1^u) & p_1^b(\xi_1^u) & \dots & p_J^b(\xi_1^u) \\
0 & \dots & 0 & 0 & \dots & 0 \\
x_1(\xi_2^{uu}) & \dots & x_J(\xi_2^{uu}) & -x_1(\xi_2^{uu}) & \dots & -x_J(\xi_2^{uu}) \\
x_1(\xi_2^{ud}) & \dots & x_J(\xi_2^{ud}) & -x_1(\xi_2^{ud}) & \dots & -x_J(\xi_2^{ud}) \\
0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0
\end{pmatrix}
\end{array}.
$$

And, we regroup the payment streams of bid and ask orders issued in event $\xi_1^d \in \Xi_1$ in a $7 \times 2J$ matrix $\hat{\phi}(\xi_1^d)$ equal to

$$
\begin{array}{c}
 \\ \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\hat{\phi}_x^a(\xi_1^d) & \dots & \hat{\phi}_J^a(\xi_1^d) & \hat{\phi}_x^b(\xi_1^d) & \dots & \hat{\phi}_J^b(\xi_1^d)
\end{array} \\
\begin{pmatrix}
0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 \\
-p_1^a(\xi_1^d) & \dots & -p_J^a(\xi_1^d) & p_1^b(\xi_1^d) & \dots & p_J^b(\xi_1^d) \\
0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 \\
x_1(\xi_2^{du}) & \dots & x_J(\xi_2^{du}) & -x_1(\xi_2^{du}) & \dots & -x_J(\xi_2^{du}) \\
x_1(\xi_2^{dd}) & \dots & x_J(\xi_2^{dd}) & -x_1(\xi_2^{dd}) & \dots & -x_J(\xi_2^{dd})
\end{pmatrix}
\end{array}.
$$

The payment matrix $\hat{\mathcal{P}}$ is a $7 \times 6J$ matrix whose columns represent the payments across all the events of a one unit trade order placed on a security at a non-terminal event

$$\begin{pmatrix} \hat{\phi}(\xi_0) & \hat{\phi}(\xi_1^u) & \hat{\phi}(\xi_1^d) \end{pmatrix}.$$

A trading strategy $b \in \mathbb{R}_+^6$ generates the payment stream $\hat{z} \in \mathbb{R}^7$ when $\hat{z} = \hat{\mathcal{P}}b$. We call payment positive span the set of payments that a trading strategy can generate. We denote it $\hat{\mathcal{B}}(p^a, p^b)$. We have

$$\hat{\mathcal{B}}(p^a, p^b) = \{\hat{z} \in \mathbb{R}^7 \mid \hat{z} = \hat{\mathcal{P}}b \text{ for some } b \in \mathbb{R}_+^{6J}\}.$$

The payment matrix is different from the *payoff matrix* which represents the payments across all *future* events of a one-unit trade order placed on a security at a non-terminal event. It is the concatenation of the payoff matrix with the period 0 payments of every order. We denote $\phi(\xi_t)$ the sub-matrix formed by selecting every rows of the matrix $\hat{\phi}(\xi_t)$ except the first. It represents the payoff streams of bid and ask orders placed in the non-terminal event

ξ_t. We denote $\mathcal{P}$ the payoff matrix. It is a $6 \times 6J$ equal to

$$\begin{pmatrix} \phi(\xi_0) & \phi(\xi_1^u) & \phi(\xi_1^d) \end{pmatrix}.$$

We denote Φ the set of columns of the payoff matrix $\mathcal{P}$.

Remark 2.1. In 2−period markets the payoff matrix only depends on securities dividends. Therefore the presence of a bid–ask spread does not modify the characterization of completeness. 2−period markets are complete if, and only if, the rank of the payoff matrix is equal to the number of states of nature, here 4.

A trading strategy $b \in \mathbb{R}^6_+$ generates the payoff stream $z \in \mathbb{R}^6$ when $z = \mathcal{P}b$. We call payoff positive span the set of payoff streams that can be generated by a trading strategy. We denote it $\mathcal{B}(p^a, p^b)$. We have

$$\mathcal{B}(p^a, p^b) = \{z \in \mathbb{R}^6 \mid z = \mathcal{P}b \text{ for some } b \in \mathbb{R}^{6J}_+\}.$$

Since a trading strategy exclusively admits non-negative coordinates, $\hat{\mathcal{B}}(p^a, p^b)$ is *the positive span* of the columns of the payment matrix $\hat{\mathcal{P}}$ and $\mathcal{B}(p^a, p^b)$ is *the positive span* of the columns of the payoff matrix $\mathcal{P}$ (hence their name). Indeed, Davis (1954) defines the positive span of a finite set of vectors $V = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ has the set p-span(V) equal to

$$\text{p-span}(V) := \{\lambda_1 v_1 + \ldots \lambda_k v_k \mid \lambda_i \geqslant 0 \text{ for all } i = 1, \ldots, k\}.$$

We say that a finite set $V \subset \mathbb{R}^n$ positively span $\mathbb{R}^n$ if p-span(V) $= \mathbb{R}^n$.

Remark 2.2. To generalize frictionless methods to markets with bid–ask spreads, it is also possible to separate a portfolio strategy between the ask portfolio strategy, consisting of the ask orders placed in the markets, and the bid portfolio strategy, consisting of the bid orders placed in the markets. However, the monotonicity of these strategies with time complicates the study since the set of payoff streams generated by a bid–ask portfolio strategy is not a positive span.

We demonstrate in the following proposition that a payoff stream is generated by a trading strategy if, and only if, a portfolio strategy exists that replicates it. Therefore, the payoff positive span is equal to the set of available payoff streams.

Proposition 2.1. The set of available payoff streams is equal to the payoff positive span.

Proof. Fix a vector $z \in \mathcal{M}(p^a, p^b)$. We are going to show $z \in \mathcal{B}(p^a, p^b)$. By assumption, there exists a portfolio strategy h such that $z(\xi) = z(h, p^a, p^b)(\xi)$ for all $\xi \in \Xi$. Recall from Equation 1 that

$$z(h, p^a, p^b)(\xi_1) = x(\xi_1)h(\xi_0) - p^b(\xi_1)\min(h(\xi_1) - h(\xi_0)), 0) - p^a(\xi_1)\max(h(\xi_1) - h(\xi_0), 0)$$

for all $\xi_1 \in \Xi_1$ and

$$z(h, p^a, p^b)(\xi_2) = x(\xi_2)h(\xi_2^b)$$

for all $\xi_2 \in \Xi_2$. Let $b \in \mathbb{R}^{6J}$ be a trading strategy associated with h that is, such that

$$\begin{cases} b^a(\xi_0) & = & \max(h(\xi_0), 0) \\ b^b(\xi_0) & = & -\min(h(\xi_0), 0) \end{cases}$$

and

$$\begin{cases} b^a(\xi_1) & = & \max(h(\xi_1) - h(\xi_0), 0) \\ b^b(\xi_1) & = & -\min\big(h(\xi_1) - h\big(\xi_0^b\big), 0\big) \end{cases}$$

for all $\xi_1 \in \Xi_2$. Note that $b^a(\xi_2)$ and $b^b(\xi_2)$ are not defined since there is no trading taking place at time 2. We have $h(\xi_0) = b^a(\xi_0) - b^b(\xi_0)$ and

$$h(\xi_1) = h(\xi_0) + b^a(\xi_1) - b^b(\xi_1)$$

for every $\xi_1 \in \Xi_1$. Hence,

$$z(\xi_1) = x(\xi_1)(b^a(\xi_0) - b^b(\xi_0)) - p^a(\xi_1)b^a(\xi_1) + p^b(\xi_1)b^b(\xi_1)$$

for all $\xi_1 \in \Xi_1$ and

$$z(\xi_2) = x(\xi_2)\Big(b^a(\xi_0) - b^b(\xi_0) + b^a(\xi_1) - b^b(\xi_1)\Big)$$

for all $\xi_2 \in \Xi_2$. Hence, we have

$$z = \phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{\xi_1 \in \Xi_1} \phi(\xi_1)\begin{pmatrix} b^a(\xi_1) \\ b^b(\xi_1) \end{pmatrix}.$$

Hence, $z = \mathcal{P}b$ and $z \in \mathcal{B}(p^a, p^b)$.

Now fix a vector $z \in \mathcal{B}(p^a, p^b)$. We are going to show that z belongs to $\mathcal{M}(p^a, p^b)$. By assumption, there exists $b \in \mathbb{R}^{6J}_+$ such that $z = \mathcal{P}b$. It is equivalent to the existence of vectors $b^a(\xi) \in \mathbb{R}^J_+$ and $b^b(\xi_t) \in \mathbb{R}^J_+$ for all $\xi \in \xi_0 \cup \Xi_1$ such that

$$\phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{\xi_1 \in \Xi_1} \phi(\xi_1)\begin{pmatrix} b^a(\xi_1) \\ b^b(\xi_1) \end{pmatrix} = z.$$

It implies the following equations

$$z(\xi_1) = x(\xi_1)(b^a(\xi_0) - b^b(\xi_0)) - p^a(\xi_1)b^a(\xi_1) + p^b(\xi_1)b^b(\xi_1)$$

for all $\xi_1 \in \Xi_1$ and

$$z(\xi_2) = x(\xi_2)\Big(b^a(\xi_0) - b^b(\xi_0) + b^a(\xi_1) - b^b(\xi_1)\Big)$$

for all $\xi_2 \in \Xi_2$. We let h be a portfolio strategy such that $h(\xi_0) = b^a(\xi_0) - b^b(\xi_0)$ and

$$\begin{cases} \max(h(\xi_1) - h(\xi_0), 0) &= b^a(\xi_1) \\ \min(h(\xi_1) - h(\xi_0), 0) &= -b^b(\xi_1) \end{cases}$$

for every $\xi_1 \in \Xi_1$. We obtain

$$z(\xi_1) = d(h(\xi_0)) - p^b(\xi_1)\min(h(\xi_1) - h(\xi_0), 0) - p^a(\xi_1)\max(h(\xi_1) - h(\xi_0), 0)$$

for all $\xi_1 \in \Xi_1$ and

$$z(\xi_2) = x(\xi_2)h(\xi_1)$$

for all terminal event $\xi_2 \in \Xi_2$. Therefore $z \in \mathcal{M}(p^a, p^b)$. $\square$

Proposition 2.1 shows that markets are dynamically complete if for every payoff stream there exists a trading strategy that generates it. Formally, markets are dynamically complete if $\mathcal{B}(p^a, p^b) = \mathbb{R}^6$.

2.5 No-arbitrage

Dynamic completeness is characterized in the frictionless case under a mild equilibrium property, the absence of arbitrage opportunity. To extend this characterization to markets with frictions, we define an arbitrage opportunity for a trading strategy instead of a portfolio strategy. We denote $\mathbb{R}^n_{++}$ the set of vectors with strictly positive coordinates and we denote $C^\star = \mathbb{R}_+ \times \mathbb{R}^k_+ \setminus 0$ the set of positive payment streams. There exists an arbitrage opportunity in the markets if there exists a trading strategy $b \in \mathbb{R}^{6J}_+$ generating a non-negative payment stream with at least one strictly positive payment, that is such that $\hat{\mathcal{P}}b > 0$. Additionally, a vector $\mu \in \mathbb{R}^7$ is said to support the markets when for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. We show in Theorem 2.1 that there is no arbitrage opportunity in the markets if, and only if, a vector of strictly positive event prices supports the markets.

Theorem 2.1. *There is no arbitrage opportunity in a frictionless market if, and only if, a vector of strictly positive event prices supports the market.*

Proof. First we assume there is no arbitrage opportunity, we are going to show that there exists a vector of strictly positive event prices such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. No-arbitrage implies there exists no trading strategy $b \in \mathbb{R}^{6J}_+$ such that $\hat{\mathcal{P}}b \in C^\star$. Therefore, we have $\hat{\mathcal{B}}(p^a, p^b) \cap C^\star = \emptyset$. In particular, let $\Delta = \{\mu \in \mathbb{R}^7_+ \mid \sum_{i=0}^6 \mu_i = 1\}$, we have $\hat{\mathcal{B}}(p^a,p^b)\cap\Delta=\emptyset$. The set $\hat{\mathcal{B}}(p^a,p^b)$ is the positive span of the columns of the payoff matrix. Hence, it is a closed convex set. Additionally, Δ is compact. Therefore the theorem of strict separation of convex applies and there exists $\mu\in\mathbb{R}^7$ such that

$$\sup_{z\in\hat{\mathcal{B}}(p^a,p^b)} z^\top\mu < \inf_{z\in\Delta} z^\top\mu.$$

Suppose that $\mu_\xi \leqslant 0$ for some event $\xi\in\Xi_0\cup\Xi_1\cup\Xi_2$. Consider $\mu'\in\Delta$ such that $\mu'_\xi=1$ and $\mu'_{\xi'}=0$ for every $\xi'\neq\xi$. Then, ${\mu'}^\top\mu\leqslant 0$ so that

$$\sup_{z\in\hat{\mathcal{B}}(p^a,p^b)} z^\top\mu < 0,$$

contradicting $z^\top\mu=0$ for $z=0$. It remains to show that $z^\top\mu\leqslant 0$ for every $z\in\hat{\mathcal{B}}(p^a,p^b)$. Suppose there exists $z'\in\hat{\mathcal{B}}(p^a,p^b)$ such that $z^\top\mu'>0$. Since $\hat{\mathcal{B}}(p^a,p^b)$ is a positive span, there exists $\alpha\in\mathbb{R}_+$ such that $\alpha z'\in\hat{\mathcal{B}}(p^a,p^b)$ and $(\alpha z')^\top\mu>\min_{z\in\Delta} z^\top\mu$, a contradiction.

Now, assume there exists $\mu\in\mathbb{R}^7_{++}$ such that for every $z\in\hat{\mathcal{B}}(p^a,p^b)$, we have $z^\top\mu\leqslant 0$. We are going to show that there is no-arbitrage opportunity. Assume by contradiction there exists a trading strategy $b\in\mathbb{R}^{6J}_+$ such that $\hat{\mathcal{P}}b\in C^\star$. Denote $\tilde{z}$ the payoff stream of this trading strategy. By assumption, we have ${z'}^\top\mu\leqslant 0$ with $\mu\in\mathbb{R}^7_{++}$ implying $z'=0$. Therefore there is no arbitrage opportunity. □

When there are frictions, supporting strictly positive event prices represents the existence of underlying no-arbitrage frictionless markets supporting the markets. The absence of arbitrage opportunity implies that strictly positive event prices support securities prices, in the sense that prices are greater than the weighted sum of expected payoffs for event prices μ, that is, we have

$$\mu_{\xi_0}p_j^a(\xi_0)\geqslant\sum_{\xi\in\Xi_1\cup\Xi_2}\mu_\xi x_j(\xi)\geqslant\mu_{\xi_0}p_j^b(\xi_0)$$

$$\mu_{\xi_1^u}p_j^a(\xi_1^u)\geqslant\mu_{\xi^{uu}}x_j(\xi^{uu})+\mu_{\xi^{ud}}x_j(\xi^{ud})\geqslant\mu_{\xi_1^u}p_j^b(\xi_1^u),$$

and

$$\mu_{\xi_1^d}p_j^a(\xi_1^d)\geqslant\mu_{\xi^{du}}x_j(\xi^{du})+\mu_{\xi^{dd}}x_j(\xi^{dd})\geqslant\mu_{\xi_1^d}p_j^b(\xi_1^d)$$

for every security $j\in J$. Therefore trading strategies permit a straightforward generalization of the fundamental theorem of asset pricing to markets with bid–ask spreads. When markets are frictionless, the payoffs stream of an ask order is the opposite of the payoff stream of the bid order on the same security in the same event. Hence, the vector of event prices supports the market with an equality sign and Theorem 2.1 coincides with the fundamental theorem of asset pricing (Harrison and Kreps (1979), Magill and Quinzii (1996)) expressed for trading strategies instead of portfolio strategies.

2.6 Characterization of Dynamic Completeness

In the absence of bid–ask spread the market presented in the previous section is dynamically complete only if at least 2 securities are traded. Proposition 2.2 shows that the presence of a bid–ask spread does not increase the number of traded securities necessary for dynamic completeness.

Proposition 2.2. Markets are dynamically complete only if at least 2 securities are traded.

Proof. By Proposition 2.1 markets are dynamically complete if $\mathcal{B}(p^a, p^b) = \mathbb{R}^6$. Then, p-span$(\Phi) = \mathbb{R}^6$. Hence as a consequence of Corollary 2.4 of Regis (2015) which states that any positive spanning set of $\mathbb{R}^6$ contains a basis of $\mathbb{R}^6$, the payoff matrix must have at least 7 columns. It implies that at least $J \geqslant \frac{7}{2(6+1-4)}$ securities must be traded. Since J takes only integer values, markets are dynamically complete only if $J \geqslant 2$. □

As for frictionless markets, having 2 securities traded each period is not sufficient for markets to be dynamically complete. Dynamic completeness also depends on the values of each security dividend and price. In frictionless markets, under no-arbitrage, markets are dynamically complete if, and only if, a unique normalized strictly positive event prices vector supports the market. We show in the following theorem that the uniqueness of this vector is equivalent to having exclusively *non-zero period*-0 *event prices* supporting the market. A vector of event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ is said to have a non-zero period-0 price if $\mu_{\xi_0} \neq 0$.

Proposition 2.3. When markets are frictionless and admit no arbitrage opportunity, a unique normalized vector of strictly positive event prices supports the market if, and only if, every non-zero supporting event prices have a non-zero period-0 price.

Proof. First, we assume that there exists a unique normalized $\nu \in \mathbb{R}^7_{++}$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. We are going to show that every supporting event prices have a non-zero period-0 price. Assume by contradiction that there exist $\mu = \begin{pmatrix} 0 \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7 \setminus \{0\}$ with $\tilde{\mu} \in \mathbb{R}^6$ such that $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\epsilon > 0$ be such that $\nu + \epsilon\mu \in \mathbb{R}^7_{++}$, then $(\nu + \epsilon\mu)^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$ contradicting the uniqueness of ν.

Now, we assume that every $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7 \setminus \{0\}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ satisfying $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$ are such that $\mu_{\xi_0} \neq 0$. We are going to show that there exists a unique normalized $\nu \in \mathbb{R}^7_{++}$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ with $\nu_{\xi_0} \in \mathbb{R}^*_+$ and $\tilde{\nu} \in \mathbb{R}^6_{++}$. Suppose by contradiction that there exists $\nu' = \begin{pmatrix} \nu'_{\xi_0} \\ \tilde{\nu}' \end{pmatrix}$ with

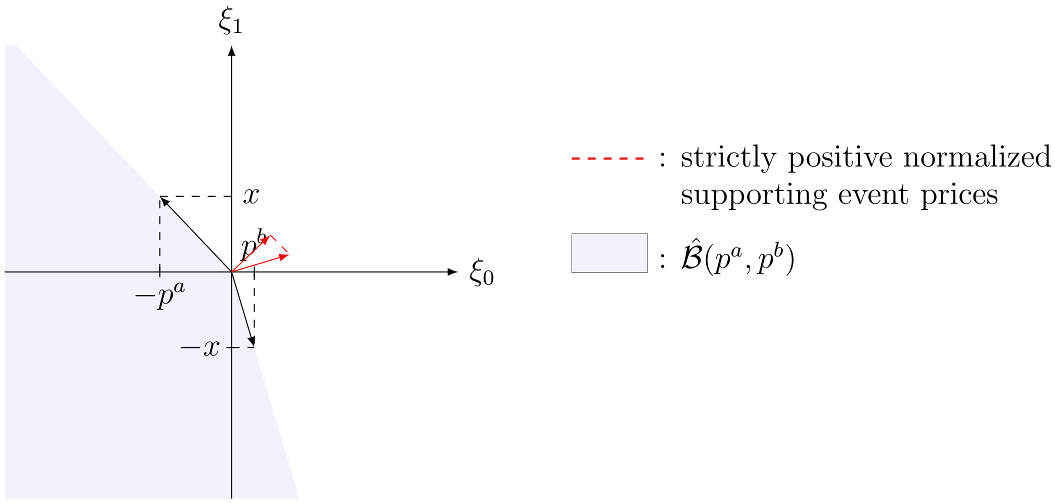


Figure 1: 2-period dynamically complete security market with no uncertainty and a single traded security x.

$\nu'_{\xi_0} \in \mathbb{R}^*_+$ and ${\nu_{\xi_0}}' \in \mathbb{R}^6_{++}$ such that $\nu' \neq \lambda\nu$ for every $\lambda \in \mathbb{R}_+$ and ${\nu'}^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$. Let $\alpha \in \mathbb{R}$ be such that $\nu_{\xi_0} = \alpha\nu'_{\xi_0}$ and $(\nu - \alpha\nu')^\top z = 0$ implying $\nu = \alpha\nu'$, a contradiction.

□

Figure 1 represents a dynamically complete (every payoff at period 1 can be traded at period 0) 2−period security market with no uncertainty and a single traded security with a bid–ask spread. The ξ_0−axis represents the payment received at period 0, and the ξ_1−axis represents the payoff received at period 1. The black vectors represent the payoff streams of bid and ask orders placed at periods 0 and 1. As illustrated in this figure, dynamic completeness does not require that a unique vector of strictly positive event prices supports the market when there are frictions. However, we demonstrate in Theorem 2.2 that every supporting event prices vector must have a non-zero period-0 price.

Theorem 2.2. *The following conditions are equivalent*

i. Markets are dynamically complete;

ii. for every payoff stream $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top\phi > 0$;

iii. every event price vector supporting the markets has a non-zero period−0 price.

Proof. The proof of *i.* equivalent to *ii.* follows from the characterization of positive spanning sets of Davis (1954). For the sake of clarity, we present it in our context. Assume that markets are dynamically complete. We are going to show that for every non-zero $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top\phi > 0$. Assume by contradiction that there exists a payoff stream $q \in \mathbb{R}^6$ such that $q^\top \phi \leqslant 0$ for every $\phi \in \Phi$. Denote ϕ_i, $i = 1, \ldots, 6J$, the i^{th} element of Φ and dynamic completeness implies that there exists a trading strategy $b \in \mathbb{R}_+^{6J}$ such that $z = \mathcal{P}b$, that is

$$z = \sum_{i=1}^{6J} b_i \phi_i$$

where b_i is the i^{th} coordinate of b. Moreover, $z^\top z > 0$ that is,

$$\sum_{i=1}^{6J} (b_i \phi_i)^\top z > 0$$

implies that at least one element of the sum is positive. Hence there exists an order payoff stream ϕ_i such that $b_i \phi_i^\top z > 0$, a contradiction. Therefore, for every non-zero $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$.

Then assume that for every payoff stream $q \in \mathbb{R}^6$ there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. We are going to show that markets are dynamically complete. Assume by contradiction that markets are dynamically incomplete. It implies that there exists a payoff stream $z \notin \mathcal{B}(p^a, p^b)$, that is $\mathcal{B}(p^a, p^b) \cap z = \emptyset$. Therefore according to Rockafellar (1970) Theorem 11.3, there exists a hyperplane containing the origin that properly separates $\mathcal{B}(p^a, p^b)$ and z. Denote q' its normal vector at the origin then for either $q = q'$ or $q = -q'$, we have $q^\top \phi \leqslant 0$ for every $\phi \in \Phi$, a contradiction.

Now, we are going to show that *ii.* is equivalent to *iii.*. First, assume that for every payoff stream $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. We are going to show that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. Let $\nu_{\xi_0} \in \mathbb{R}$, $\tilde{\nu} \in \mathbb{R}^6$ and let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ be such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. By assumption, there exists $\phi \in \Phi$ such that $\phi^\top \tilde{\nu} > 0$. It implies $\nu_{\xi_0} \neq 0$.

Then, we assume that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. We are going to show that for every payoff stream $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. Assume by contradiction that there exists $\tilde{\nu} \in \mathbb{R}^6$, such that $\phi^\top \tilde{\nu} \leqslant 0$ for every $\phi \in \Phi$. Therefore, we have $\nu = \begin{pmatrix} 0 \\ \tilde{\nu} \end{pmatrix} \in \mathbb{R}^7$ such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, a contradiction.

□

We illustrate Theorem 2.2 in a simple 3−period market with no uncertainty in Figure

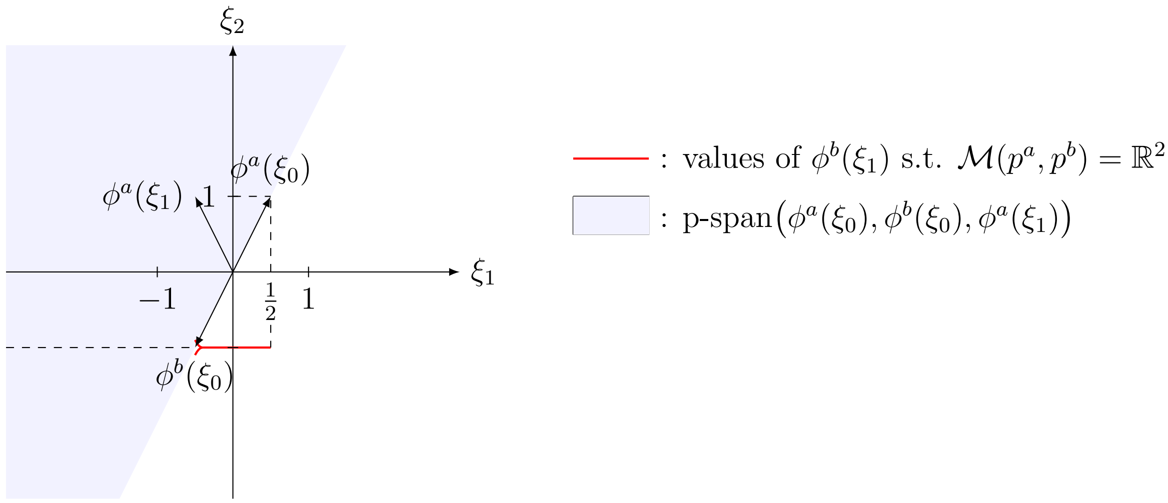


Figure 2: Geometry of a 3-period market with no uncertainty, no arbitrage opportunity, and a single security $x_j = (0.5, 1)$ *(that is,* $S = 1$ *and* $J = 1$*) with ask price equal to* 0.5 *at period* 1*.*

2 in Section 2.7. We provide an example of dynamically complete markets with bid–ask spreads in Section 2.8.

2.7 Geometric Representation of Dynamic Completeness

We consider a 3−period security market with no arbitrage opportunity, one security traded at each period, and no uncertainty about the future. Figure 2 represents the payoff streams of bid and ask orders placed at periods 0 and 1. It depicts the case in which the security's dividends x is equal to $(0.5, 1)$, its ask price at period 1 is equal to 0.5, and its bid price, $p_j^b(\xi_1)$, is not specified. The ξ_1−axis represents the payoff received at period 1, and the ξ_2−axis represents the payoff received at period 2. The vectors represent the payoff streams of the security. In addition, we represent in the following graph the unfolding of the dividends and prices of x.

$$j \;—\!— \; (0.5, 0.5, p_j^b(\xi_1)) \;—\!—\; (1, 0, 0)$$

The blue set represents the positive span of the payoff streams of bid and ask orders placed at period 0 and ask orders placed at period 1. We remark that markets are dynamically complete if, and only if, the payoff stream of a bid order placed at period 1 is not included in this set. Additionally, by no-arbitrage, p^b cannot be strictly greater than p^a. Since the second coordinate of $\phi^b(\xi_1)$ is fixed (it is equal to the dividend of j at period 2), it implies that markets are dynamically complete if, and only if, $\phi^b(\xi_1)$ takes a value on the red line. Hence, markets are dynamic complete if, and only if, $-0.5 < p^b(\xi_1) \leqslant p^a(\xi_1)$. When negative values of $p^a(\xi_1)$ and $p^b(\xi_1)$ are not economically meaningful we can directly conclude that as for frictionless markets, dynamic completeness is equivalent to having the security's price at period 1 is different from the dividend received in this event $p^b(\xi_1) \neq -x(\xi_1)$ and $p^a(\xi_1) \neq x(\xi_1)$. More generally, it follows from Theorem 2.2 that a 3-period market with no uncertainty and a single security j is dynamically complete if, and only if, for every $q \in \mathbb{R}^2$ such that $q^\top x$ is equal to zero, the product of q with the ask orders payoff streams and the product of q with the bid orders' payoff stream placed at period 1 have strictly opposite signs.

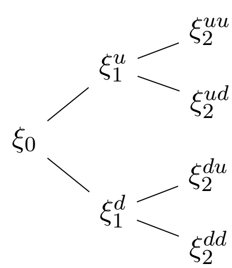
Proposition 2.4. 3−period markets with no uncertainty and a single security are dynamically complete if, and only if, for every payoff stream $q \in \mathbb{R}^2$ such that $q^\top \phi^a(\xi_0) = 0$, $q^\top \phi^a(\xi_1)$ and $q^\top \phi^b(\xi_1)$ have strictly opposite sign.

In the following subsection, we present examples of 3−period dynamically complete markets with bid–ask spreads.

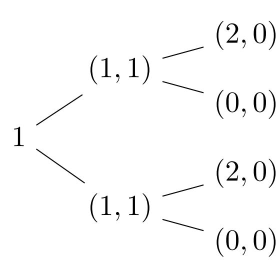
2.8 Examples

The presence of a bid–ask spread does not necessarily result in dynamic incompleteness. In the following example, we present 3−period frictionless and dynamically incomplete security markets that become dynamically complete when a market maker charges a transaction cost to compensate her services.

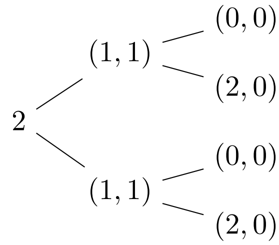
Example 2.1. We consider the following 3-period market with uncertainty at period 0 regarding the outcome of future periods.



There are two securities available for trading at period 0 and period 1. Their dividends are equal to $(x_1, x_2) \in \mathbb{R}^6 \times \mathbb{R}^6$. We initially assume the securities' prices present no bid–ask spread. The following graph presents the unfolding of security 1 dividends and prices.



We present the payoffs of security 2 similarly.

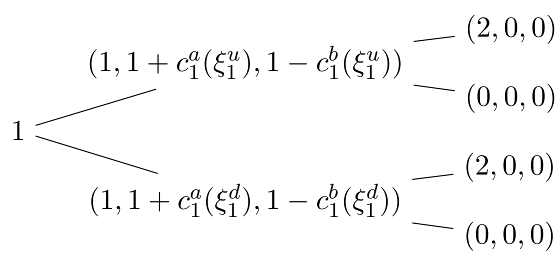


The one-period matrix in event ξ_0 is equal to

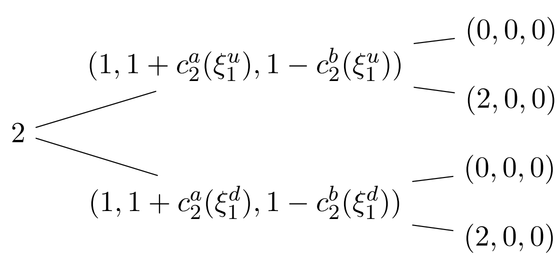
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

It is of rank 1 therefore markets are dynamically incomplete.

Now, we assume securities dividends are unchanged, but the services of the market makers are costly. In each event $\xi_1 \in \Xi_1$, they charge a transaction cost $0 < c_j^a(\xi_1) < p_j(\xi_1)$ on ask orders on j and a transaction cost $0 < c_j^b(\xi_1) < p_j(\xi_1)$ on bid orders on j. Hence, the new ask price of security j is equal to $p_j^a(\xi_1) = p_j(\xi_1) + c_j^a(\xi_1)$ and its new bid price in each event ξ_1 is equal to $p_j^b(\xi_1) = p_j(\xi_1) - c_j^b(\xi_1)$. We present the payoffs of j in a tree in which a triplet $\left(x_j(\xi), p_j^a(\xi), p_j^b(\xi)\right)$ representing the dividend of j, its ask price and its bid price in event ξ is associated to each non-initial node.



We present the payoffs of security 2 similarly.



The payoff matrix is equal to

$$
\begin{array}{c}
\\
\xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
x_1 & x_2 & -x_1 & -x_2 & \phi_1^a(\xi_1) & \phi_2^a(\xi_1) & \phi_1^b(\xi_1) & \phi_2^b(\xi_1) & \phi_1^a(\xi_1^d) & \phi_2^a(\xi_1^d) & \phi_1^b(\xi_1^d) & \phi_2^b(\xi_1^d)
\end{array}\\
\begin{pmatrix}
1 & 1 & -1 & -1 & -(1+c_1^a(\xi_1^u)) & -(1+c_2^a(\xi_1^u)) & 1-c_1^b(\xi_1^u) & 1-c_2^b(\xi_1^u) & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -(1+c_1^a(\xi_1^d)) & -(1+c_2^a(\xi_1^d)) & 1-c_1^b(\xi_1^d) & 1-c_2^b(\xi_1^d) \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2
\end{pmatrix}
\end{array}.
$$

We are going to show that markets are dynamically complete. Assume by contradiction that there exists a non-zero $z \in \mathbb{R}^k$ such that $z\phi \leqslant 0$ for all $\phi \in \Phi$. Denote z_i the i^{th} coordinate of z. We have $zx_j \leqslant 0$ and $-zx_j \leqslant 0$ which imply

$$z_1 + z_2 + 2z_3 + 2z_5 = 0 \quad \text{and} \quad z_1 + z_2 + 2z_4 + 2z_6 = 0.$$

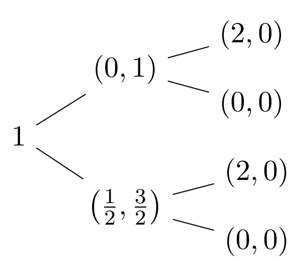
Inequalities $z\phi_1^a(\xi_1^u) \leqslant 0$ and $z\phi_1^b(\xi_1^u) \leqslant 0$ imply

$$-(1 + c_1^a(\xi_1^u))z_1 + 2z_3 \leqslant 0 \quad \text{and} \quad (1 - c_1^b(\xi_1^u))z_1 - 2z_3 \leqslant 0.$$

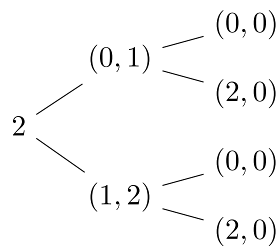
Therefore, we have $z_1 \geqslant 0$ and $z_3 \geqslant 0$. Similarly, we obtain from the other inequalities $z_i \geqslant 0$ for $i = 2, 4, 5, 6$. Hence $z = 0$, a contradiction. We conclude that there does not exist a non-zero z such that $z\phi \leqslant 0$ for all $\phi \in \Phi$. Markets are dynamically complete.

Now, we present an example of 3−period security markets which becomes dynamically incomplete when the market maker services become costly.

Example 2.2. We assume there are 2 traded securities available for trading. Their dividends are equal to $(x_1, x_2) \in \mathbb{R}^6 \times \mathbb{R}^6$. We initially assume there is no bid–ask spread, securities can be purchased and sold at a same price $p_j(\xi)$ in every event $\xi \in \Xi_0 \cup \Xi_1 \cup \Xi_2$. We represent the unfolding of the dividends and prices of j in a tree by associating a couple $(x_j(\xi), p_j(\xi))$ to each event $\xi \in \Xi_1 \cup \Xi_2$.



We present the payoffs of security 2 similarly.



The one-period payoff matrix in event ξ_0 is equal to

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

The one-period payoff matrices in events $\xi_1 \in \Xi_1$ are equal to

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

They are all of rank 2 therefore markets are dynamically complete.

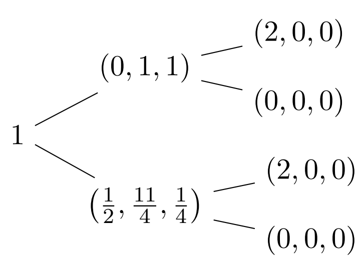
Now, we assume the market makers services are costly in event ξ_1^d, they charge a transaction cost $p_j(\xi_1^d) > c_j(\xi_1^d) > 0$ on security j such that

$$p_1^a(\xi_1^d) + x_1(\xi_1^d) = p_2^a(\xi_1^d) + x_2(\xi_1^d)$$

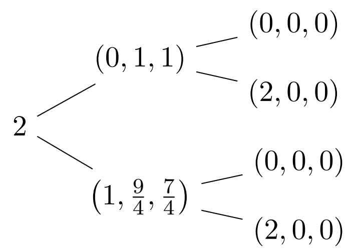
where $p_1^a(\xi_1^d) = p_1(\xi_1^d) + c_1(\xi_1^d)$ and $p_2^a(\xi_1^d) = p_2(\xi_1^d) + c_2(\xi_1^d)$. We have

$$c_1(\xi_1^d) - c_2(\xi_1^d) = 1.$$

We take for example $c_1(\xi_1^d) = \frac{5}{4}$ and $c_1(\xi_1^d) = \frac{1}{4}$. We present the unfolding of the dividends and prices of x_1 in the following tree.



We present them similarly for security 2.



The payoff matrix is equal to

$$
\begin{array}{c} \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} x_1 & x_2 & -x_1 & -x_2 & \phi_1^a(\xi_1^u) & \phi_2^a(\xi_1^u) & \phi_1^b(\xi_1^u) & \phi_2^b(\xi_1^u) & \phi_1^a(\xi_1^d) & \phi_2^a(\xi_1^d) & \phi_1^b(\xi_1^d) & \phi_2^b(\xi_1^d) \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 & -\frac{11}{4} & -\frac{9}{4} & \frac{1}{4} & \frac{7}{4} \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2
\end{pmatrix}
\end{array}.
$$

Let $z = (-3, 1, -\frac{3}{2}, -\frac{3}{2}, \frac{5}{4}, 1)$. The product of z with any of the columns of the payoff matrix is negative. Hence, this payoff does not belong to the set of available payoff streams. It cannot be replicated by a dynamic trading strategy using the securities 1 and 2. The bid–ask spread makes the markets dynamically incomplete.

3 Multi-period Markets With Bid-Ask Spreads

This section presents the characterization of dynamic completeness in general multi-period security markets with frictions creating a bid–ask spread. We assume there is no restriction to short-selling and no limitation to the quantity of security purchased and sold at the initial period. We also assume agents can infinitely split their orders, and they share the same information structure. Hence incompleteness does not result from the presence of one of these frictions.

3.1 Uncertainty And Information

The future is uncertain. We use the same notations as LeRoy and Werner (2014). Uncertainty is specified by a set of states S. Each of the states represents a description of the economic environment for all periods $t = 0, 1, \ldots, T$. At period 0 agents do not know which state will be realized. However, as time passes, they obtain more and more information about the state. At period T they discover the actual state. Formally, the information of agents at period t is described by a partition F_t of the set of states S (a partition F_t of S represents a collection of subsets of S such that each state s belongs to exactly one element of F_t). The period-0 partition is the trivial partition $F_0 = \{S\}$. The

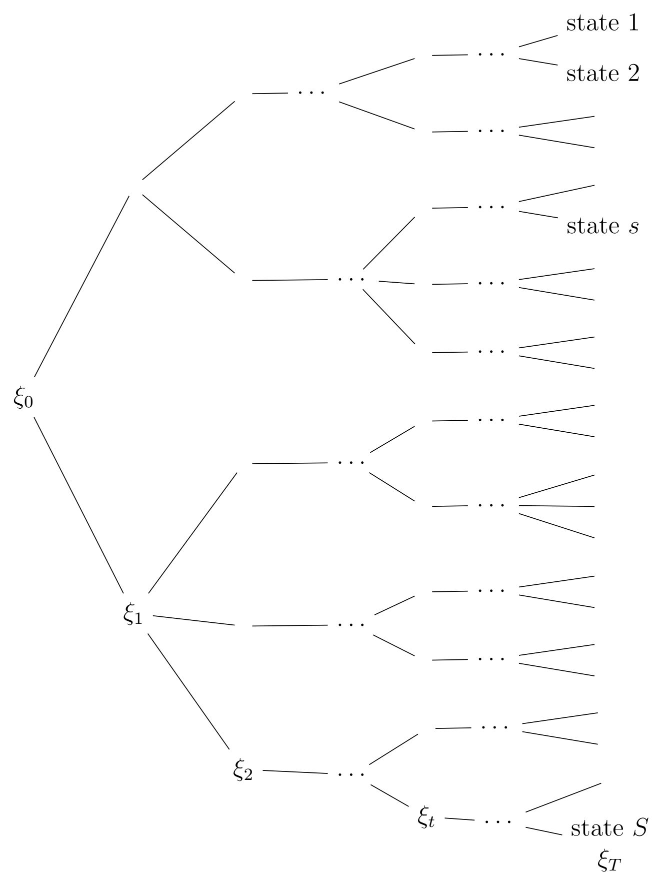


Figure 3: Example of an event tree with T periods and S states of the nature.

period$-T$ partition is the total partition $F_T = \{\{s\} : s \in S\}$. The partition F_{t+1} is finer than partition F_t; that is, the element of period$-(t+1)$ partition to which it belongs. The $(T+1)$-tuple of partitions $\{F_0, F_1, \dots, F_T\}$ is the information filtration F. The partitions are assumed to be common across agents; that is, all agents possess the same information. The number k_t denotes the number of elements in the filtration F_t.

For better exposition, we represent the information filtration as an event tree with each element of partition F_t being a period$-t$ event denoted ξ_t. An event is a node of the event tree (see Figure 3). The event $\xi_0 = F_0$ represents the root node. We denote ξ_t^{++} the set of successors of the event ξ_t. It is equal to the set of events $\xi_\tau \subset \xi_t$ with $\xi_\tau \in F_\tau$ for $\tau > t$. The immediate successors of ξ_t are the events $\xi_{t+1} \subset \xi_t$ with $\xi_{t+1} \in F_{t+1}$. The number of immediate successor of event ξ_t is denoted $k(\xi_t)$. The predecessor of the event ξ_t are the events $\xi_\tau \supset \xi_t$ with $\xi_\tau \in F_\tau$ for $\tau < t$. The unique immediate predecessor of ξ_t is the event $\xi_{t-1} \subset \xi_t$. It is denoted ξ_t^-. The set of all events at all future periods $t = 1, \dots, T$ is denoted Ξ, and $k = \#(\Xi)$ represents the number of future events, that is events in Ξ. Therefore there is a total of $k+1$ events including ξ_0.

3.2 Securities, Portfolios And Payoffs

We consider security markets with J securities traded at each period up to period T-1. Each security is characterized by the dividends it pays at each period, a vector x_j of $\mathbb{R}^k$. The dividend matrix $X = \begin{pmatrix} x_1 & \dots & x_J \end{pmatrix}$ represents the dividend streams of the J securities traded in the markets. The dividend of a security j in an event $\xi \in \Xi$ is denoted $x_j(\xi)$. We gather the dividends on every security in an event $\xi \in \Xi$, in a single row vector $x(\xi) = (x_1(\xi), \dots, x_J(\xi))$.

We consider a market with two types of actors investors and market makers. An investor represents any party that trades on a financial security market. A market maker is a party who actively quotes two-sided markets in a particular security, providing bids and asks. The bid–ask spread compensates the market maker services. Investors purchase and sell securities to a market maker. They buy at the ask price and sell at the bid price (a market maker buys at the bid price and sells at the ask price). We adopt their perspective in the following. The ask price of security j in an event $\xi \in \Xi$ is denoted $p^a_j(\xi) \in \mathbb{R}^J$ and its bid price is denoted $p^b_j(\xi) \in \mathbb{R}^J$. For notational convenience, we have period$-T$ bid and ask prices $p^b_j(\xi_T) \in \mathbb{R}^J$ and $p^a_j(\xi_T) \in \mathbb{R}^J$ even though trading does not take place at period T. These prices are all set equal to zero. We denote $p^a(\xi) \in \mathbb{R}^J$ the row vector with coordinates equal to the J securities' ask price in the event ξ. Similarly, we denote $p^b(\xi) \in \mathbb{R}^J$ the row vector with coordinates equal to the J securities' bid price in the event ξ.

At each non-terminal period, investors constitute a portfolio of securities by trading on the markets. A portfolio of securities in an event $\xi \in \xi_0 \cup \Xi$ is represented as a vector $h(\xi) \in \mathbb{R}^J$. The j^{th} coordinate of $h(\xi)$ is denoted $h_j(\xi)$, it represents the holding of security j. If it is positive, then the portfolio holder is entitled to receive the dividends of security j in the successor events. If it is negative, she is entitled to pay the dividends of j to her counterpart in the successor events. We denote h_t the portfolios held in every event of period t. The $T-$uplet $(h_0, h_1, \dots, h_{T-1})$ is called a portfolio strategy.

3.3 Dynamic Completeness and the Set of Available Payoff Streams

The payoff of a portfolio strategy h in event ξ_t is denoted $z(h, p^a, p^b)(\xi_t)$. An investor first receives the dividends of the portfolio she had constituted in the previous period. Then she trades on the markets. Hence her payoff is equal to

$$\overbrace{x(\xi_t)h(\xi_t^-)}^{\text{dividends}} - \overbrace{p^b(\xi_t)\min\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big)}^{\text{sales revenue}} - \overbrace{p^a(\xi_t)\max\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big)}^{\text{purchases cost}}. \tag{2}$$

Therefore the payoff equals the magnitude of the payment at ξ_t to the investor (or, if negative, from the investor). We denote $z_t(h, p^a, p^b)$ the vector of payoffs $z(h, p^a, p^b)(\xi_t)$ in all period$-t$ events. We say that a portfolio strategy h replicates a payoff stream $z \in \mathbb{R}^k$ if $z(\xi) = z(h, p^a, p^b)(\xi)$ for all $\xi \in \Xi$.

Markets with bid–ask spreads are dynamically complete if it is possible to construct for every payoff stream a portfolio strategy that replicates it. The set of payoff streams available via trades on securities is the set $\mathcal{M}(p^a, p^b)$ equal to

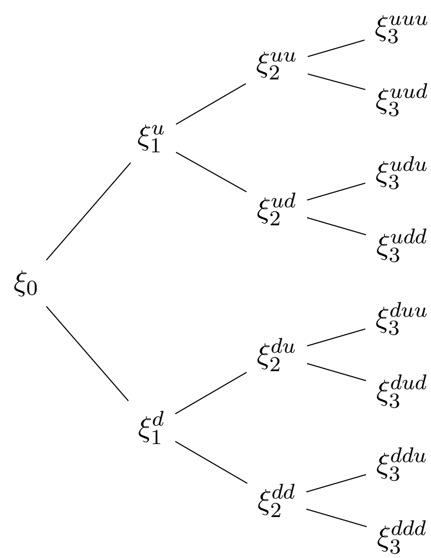
$$\left\{z \in \mathbb{R}^k \mid \exists h \text{ s. t. } z(\xi) = z(h, p^a, p^b)(\xi) \text{ for all } \xi \in \Xi\right\}.$$

In frictionless security markets, this set is called the asset span since it is equal to the span of the columns of the market matrix. In markets with bid–ask spreads this set is not a span. We, therefore, call it the set of available payoff streams. Markets with bid–ask spreads are dynamically complete if $\mathcal{M}(p^a, p^b) = \mathbb{R}^k$, otherwise they are dynamically incomplete. We use the notation $\mathcal{M}(p^a, p^b)$ to emphasize the fact that the presence of a bid–ask spread affects dynamic completeness.

3.4 Trading Strategy

First, we introduce the concept of trading strategies. A trading strategy records the unfolding of market orders placed in each event. We denote $b^a(\xi) \in \mathbb{R}^J_+$ the ask orders placed in a non-terminal event ξ and $b^b(\xi) \in \mathbb{R}^J_+$ the bid orders placed in a non-terminal event ξ. We emphasize that orders exclusively admit non-negative values as opposed to portfolios that equally admit negative values. An order placed on a security entitles its issuer to a stream of payoffs in the following periods. Previous to defining the payoff stream of an order, we introduce the necessary notations. We denote $\mathbb{1}_{\xi_t} \in \mathbb{R}^{k+1}$ the vector with coefficient 1 for the coordinate corresponding to the event ξ_t and 0 in all other events and we denote $\mathbb{1}_{\xi_t^{++}} \in \mathbb{R}^{k+1}$ the vectors with coefficient 1 in all coordinates corresponding to an event $\xi_\tau \subset \xi_t$ and 0 in all other events, that is $\mathbb{1}_{\xi_t^{++}} \in \mathbb{R}^{k+1}$ takes the value 1 in all successor events of event ξ_t and 0 otherwise. We illustrate this notation in the following example.

Example 3.1. Consider the following 4-period market



We have $\mathbb{1}_{\xi_1^{u++}} = \begin{pmatrix} 0\\0\\0\\1\\1\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0 \end{pmatrix}$ and $\mathbb{1}_{\xi_1^{d++}} = \begin{pmatrix} 0\\0\\0\\0\\0\\1\\1\\0\\0\\0\\0\\1\\1\\1\\1 \end{pmatrix}$

We define the function

$$\begin{array}{cccc} f & : & \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} & \to & \mathbb{R}^{k+1} \\ & & ((x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k)) & \mapsto & (x_1 y_1, x_2 y_2, \dots, x_k y_k) \end{array}$$

which associates the product of coordinates to two vectors. We denote $x_j\big(\xi_t^{++}\big)$ the dividends of a unit of security j purchased (or sold) in an event ξ_t, $0 < t < T$. We have $x_j\big(\xi_t^{++}\big) = f\Big(x_j, \mathbb{1}_{\xi_t{}^{++}}\Big)$.

The payment stream of an ask order placed on security j in an event $\xi_t \in F_t$, $0 \leqslant t < T$, is represented by the vector $\hat{\phi}_j^a(\xi_t)$ with coordinates equal to the ask price in event ξ_t, the dividends associated with the holding of the security in successors of event ξ_t and zero in all other events, that is,

$$\hat{\phi}_j^a(\xi_t) = -p_j^a(\xi_t)\mathbb{1}_{\xi_t} + x_j\big(\xi_t^{++}\big).$$

The payment stream of a bid order placed on security j in an event $\xi_t \in F_t$, $0 \leqslant t < T$, is represented by the vector $\hat{\phi}_j^b(\xi_t)$ with coordinates equal to the bid price in event ξ_t, the dividends due in successors of event ξ_t and zero in all other events, that is,

$$\hat{\phi}_j^b(\xi_t) = p_j^b(\xi_t)\mathbb{1}_{\xi_t} - x_j\big(\xi_t^{++}\big).$$

The payment streams of bid and ask orders placed in event ξ_t on the J securities is a $k+1 \times 2J$ matrix denoted $\hat{\phi}(\xi_t)$ with entries $\hat{\phi}^a_j(\xi_t)$, and $\hat{\phi}^b_j(\xi_t)$ for all $1 \leqslant j \leqslant J$ that is,

$$\hat{\phi}(\xi_t) = \begin{pmatrix} \hat{\phi}^a_1(\xi_t) & \dots & \hat{\phi}^a_J(\xi_t) & \hat{\phi}^b_1(\xi_t) & \dots & \hat{\phi}^b_J(\xi_t) \end{pmatrix}.$$

We denote $\hat{\Phi}$ the set of all the payment streams. We have $\#\hat{\Phi} = 2J(k+1-S)$. Therefore, a trading strategy is a vector of $\mathbb{R}^{2J(k+1-S)}_+$. Its coordinates represent the size of the buy and bid orders placed on the markets in each trading event. The payment matrix $\hat{\mathcal{P}}$ is a $k+1 \times 2J(k+1-S)$ matrix with entries $\hat{\phi} \in \hat{\Phi}$ such that

$$\hat{\mathcal{P}} = \begin{pmatrix} \hat{\phi}(\xi_0) & \hat{\phi}(\xi_1(1)) & \dots & \hat{\phi}(\xi_1(k_1)) & \dots & \hat{\phi}(\xi_{T-1}(1)) & \dots & \hat{\phi}(\xi_{T-1}(k_{T-1})) \end{pmatrix}.$$

The first J columns of the payment matrix represent the payment streams of ask orders placed at period 0. The columns $J+1$ to $2J$ of the payment matrix represent the payment streams of bid orders placed at the period 0. The successor columns represent the payment streams of ask orders placed at period $1 \leqslant t \leqslant T-1$ in some event at period t. We presented examples of payment matrix in Section 2.

The payment positive span $\hat{\mathcal{B}}(p^a, p^b)$ is the set of payment streams that can be generated by a trading strategy, that is

$$\hat{\mathcal{B}}(p^a, p^b) = \left\{ \hat{z} \in \mathbb{R}^k \mid \hat{z} = \hat{\mathcal{P}} b \text{ for some } b \in \mathbb{R}^{2J(k+1-S)}_+ \right\}.$$

Since trading strategies exclusively accept non-negative coordinates, the payment positive span is the positive span in the mathematical sense (see Section 2 for a definition) of the set of payment streams.

The payment matrix is different from the *payoff matrix* which represents the payments across all *future* events of a one-unit trade order placed on a security at a non-terminal event. It is the concatenation of the payoff matrix with the period 0 payments of every order. We denote $\phi(\xi_t)$ the sub-matrix formed by selecting every rows of the matrix $\hat{\phi}(\xi_t)$ except the first. It represents the payoff streams of bid and ask orders placed in the non-terminal event ξ_t. We denote $\mathcal{P}$ the payoff matrix. It is a $k \times 2J(k+1-S)$ equal to

$$\begin{pmatrix} \phi(\xi_0) & \phi(\xi_1(1)) & \dots & \phi(\xi_1(k_1)) & \dots & \phi(\xi_{T-1}(1)) & \dots & \phi(\xi_{T-1}(k_{T-1})) \end{pmatrix}.$$

We denote Φ the set of columns of the payoff matrix $\mathcal{P}$.

The payoff positive span $\mathcal{B}(p^a, p^b)$ is the set of payoff streams that can be realized by a trading strategy, that is

$$\mathcal{B}(p^a, p^b) = \left\{ z \in \mathbb{R}^k \mid z = \mathcal{P} b \text{ for some } b \in \mathbb{R}^{2J(k+1-S)}_+ \right\}.$$

Since trading strategies only take positive values, the payoff positive span is equal to the positive span (see Section 2 for a definition) of the set of columns of the payoff matrix that is,

$$\mathcal{B}(p^a, p^b) = \text{p-span}(\Phi).$$

A trading strategy generates a payoff stream if, and only if, a portfolio replicates it as well. Put differently, the set of available payoff streams is equal to the payoff positive span.

Proposition 3.1. The set of available payoff streams is equal to the payoff positive span.

Proof. Fix a vector $z \in \mathcal{M}(p^a, p^b)$. We are going to show that $z \in \mathcal{B}(p^a, p^b)$. By assumption, there exists a portfolio strategy h such that $z_t = z_t(h, p^a, p^b)$ for all $1 \leqslant t \leqslant T$. Recall from Equation 2 that

$$z(h, p^a, p^b)(\xi_t) = d\big(h\big(\xi_t^-\big)\big) - p^b(\xi_t)\min\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big) - p^a(\xi_t)\max\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big).$$

that is, period t payoff in event ξ_t is equal to the dividend received from holding the portfolio $h(\xi_t^b)$ at the beginning of period t plus the gain earned from trading taking place at period t. Let $b \in \mathbb{R}^{2J(k+1-S)}$ be a trading strategy such that

$$\begin{cases} b^a(\xi_0) &= \max(h(\xi_0), 0) \\ b^b(\xi_0) &= -\min(h(\xi_0), 0) \\ b^a(\xi_t) &= \max\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big) \text{ for all } 0 < t < T \\ b^b(\xi_t) &= -\min\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big) \text{ for all } 0 < t < T \end{cases}.$$

We denote $E(\xi_t)$ the set of predecessor of ξ_t (see Section 3.1). We have

$$h(\xi_t) = \sum_{\xi_\tau \in E(\xi_t)} \Big(b^a(\xi_\tau) - b^b(\xi_\tau)\Big)$$

for every $0 \leqslant t \leqslant T - 1$. Hence,

$$z(h, p^a, p^b)(\xi_t) = x(\xi_t) \sum_{\xi_\tau \in E(\xi_t)} \Big(b^a(\xi_\tau) - b^b(\xi_\tau)\Big) - p^a(\xi_t)b^a(\xi_t) + p^b(\xi_t)b^b(\xi_t)$$

for all $\xi_t \in F_t$ and all $0 < t < T$ and

$$z(h, p^a, p^b)(\xi_T) = x(\xi_T) \sum_{\xi_\tau \in E(\xi_t)} \Big(b^a(\xi_\tau) - b^b(\xi_\tau)\Big)$$

for all $\xi_T \in F_T$ where $b^a(\xi_t)$ represents the quantities of securities purchased in event ξ_t, $b^b(\xi_t)$ represents the quantities of securities sold in event ξ_t and $\sum_{\xi_\tau \in E(\xi_t)}\big(b^a(\xi_\tau) - b^b(\xi_\tau)\big)$ are the cumulative quantities of securities traded up to time t. Note that b_T is not defined since there is no trading taking place at time T. Hence, we have

$$\phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{t=1}^{T-1}\sum_{\xi_t\in F_t}\phi(\xi_t)\begin{pmatrix} b^a(\xi_t) \\ b^b(\xi_t) \end{pmatrix} = z$$

which is equal to $\mathcal{P}b = z$. Therefore, $z \in \mathcal{B}(p^a, p^b)$.

Now fix a vector $z \in \mathcal{B}(p^a, p^b)$. We are going to show that z belongs to $\mathcal{M}(p^a, p^b)$. By assumption, there exists $b \in \mathbb{R}_+^{2J(k+1-S)}$ such that $z = \mathcal{P}b$. It is equivalent to the existence of row vectors $b^a(\xi_t) \in \mathbb{R}^J$ and $b^b(\xi_t) \in \mathbb{R}^J$ for all $\xi_t \in F_t$ and all $0 \leqslant t \leqslant T-1$ such that

$$\phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{t=1}^{T-1}\sum_{\xi_t\in F_t}\phi(\xi_t)\begin{pmatrix} b^a(\xi_t) \\ b^b(\xi_t) \end{pmatrix} = z.$$

It implies the following equality

$$x(\xi_t)\sum_{\xi_\tau\in E(\xi_t)}\Big(b^a(\xi_\tau) - b^b(\xi_\tau)\Big) - p^a(\xi_t)b^a(\xi_t) + p^b(\xi_t)b^b(\xi_t) = z(\xi_t)$$

for all non-terminal event ξ_t. We let h be a portfolio strategy such that

$$\begin{cases} h(\xi_t^-) & = & \sum_{\xi_\tau\in E(\xi_t)}\big(b^a(\xi_\tau) - b^b(\xi_\tau)\big) \\ \max\big(h(\xi_t) - h(\xi_t^-), 0\big) & = & b^a(\xi_t) \\ \min\big(h(\xi_t) - h(\xi_t^-), 0\big) & = & -b^b(\xi_t). \end{cases}$$

We obtain

$$z(\xi_t) = d\big(h\big(\xi_t^-\big)\big) - p^b(\xi_t)\min\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big) - p^a(\xi_t)\max\big(h(\xi_t) - h\big(\xi_t^-\big), 0\big)$$

for all non-terminal event ξ_t and

$$z(\xi_T) = x(\xi_T)h(\xi_T^b)$$

for all terminal event ξ_T. Therefore $z \in \mathcal{M}(p^a, p^b)$. □

Proposition 3.1 implies that the set of available payoff streams is equal to the payoff positive span. Therefore, a market is dynamically complete when $\mathcal{B}(p^a, p^b) = \mathbb{R}^k$ and trading strategies can be used to characterize dynamic completeness.

3.5 No-arbitrage

Dynamic completeness is characterized in the frictionless case under a mild equilibrium property, the absence of arbitrage opportunity. To extend this characterization to markets with frictions, we define an arbitrage opportunity for a trading strategy instead of a portfolio strategy. We denote $\mathbb{R}^n_{++}$ the set of vectors with strictly positive coordinates and we denote $C^\star = \mathbb{R}_+ \times \mathbb{R}^k_+ \setminus 0$ the set of positive payment streams. There exists an arbitrage opportunity in the markets if there exists a trading strategy $b \in \mathbb{R}_+^{2J(k+1-S)}$ generating a non-negative payment stream with at least one strictly positive payment, that is such that $\hat{\mathcal{P}}b > 0$. Additionally, a vector $\mu \in \mathbb{R}^{k+1}$ is said to support the markets when for every $\hat{z} \in \hat{\mathcal{B}}(p^a, p^b)$, we have $\hat{z}^\top \mu \leqslant 0$. We show in Theorem 3.1 that there is no arbitrage opportunity in the markets if, and only if, a vector of strictly positive event prices supports the markets.

Theorem 3.1. *There is no arbitrage opportunity in a frictionless market if, and only if, a vector of strictly positive event prices supports the market.*

Proof. First we assume there is no arbitrage opportunity, we are going to show that there exists a vector of strictly positive event prices such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. No-arbitrage implies there exists no trading strategy $b \in \mathbb{R}_+^{2J(k+1-S)}$ such that $\hat{\mathcal{P}}b \in C^\star$. Therefore, we have $\hat{\mathcal{B}}(p^a, p^b) \cap C^\star = \emptyset$. In particular, let $\Delta = \{\mu \in \mathbb{R}_+^{k+1} \mid \sum_{i=0}^k \mu_i = 1\}$, we have $\hat{\mathcal{B}}(p^a, p^b) \cap \Delta = \emptyset$. The set $\hat{\mathcal{B}}(p^a, p^b)$ is the positive span of the columns of the payoff matrix. Hence, it is a closed convex set. Additionally, Δ is compact. Therefore the theorem of strict separation of convex applies and there exists $\mu \in \mathbb{R}^{k+1}$ such that

$$\sup_{z \in \hat{\mathcal{B}}(p^a, p^b)} z^\top \mu < \inf_{z \in \Delta} z^\top \mu.$$

Suppose that $\mu_\xi \leqslant 0$ for some event $\xi \in \Xi \cup \xi_0$. Consider $\mu' \in \Delta$ such that $\mu'_\xi = 1$ and $\mu'_{\xi'} = 0$ for every $\xi' \neq \xi$. Then, ${\mu'}^\top \mu \leqslant 0$ so that

$$\sup_{z \in \hat{\mathcal{B}}(p^a, p^b)} z^\top \mu < 0,$$

contradicting $z^\top \mu = 0$ for $z = 0$. It remains to show that $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Suppose there exists $z' \in \hat{\mathcal{B}}(p^a, p^b)$ such that $z^\top \mu' > 0$. Since $\hat{\mathcal{B}}(p^a, p^b)$ is a positive span, there exists $\alpha \in \mathbb{R}_+$ such that $\alpha z' \in \hat{\mathcal{B}}(p^a, p^b)$ and $(\alpha z')^\top \mu > \min_{z \in \Delta} z^\top \mu$, a contradiction.

Now, assume there exists $\mu \in \mathbb{R}_{++}^{k+1}$ such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. We are going to show that there is no-arbitrage opportunity. Assume by contradiction there exists a trading strategy $b \in \mathbb{R}_+^{2J(k+1-S)}$ such that $\hat{\mathcal{P}}b \in C^\star$. Denote $\tilde{z}$ the payoff stream of this trading strategy. By assumption, we have ${z'}^\top \mu \leqslant 0$ with $\mu \in \mathbb{R}_{++}^{k+1}$ implying $z' = 0$. Therefore there is no arbitrage opportunity. □

When there are frictions, supporting strictly positive event prices represents the existence of underlying no-arbitrage frictionless markets supporting the markets. The absence of arbitrage opportunity implies that strictly positive event prices support securities prices, in the sense that prices are greater than the weighted sum of expected payoffs for event prices μ, that is, we have

$$\mu_{\xi_t} p_j^a(\xi_t) \geqslant \sum_{\xi \in \xi_t^{++}} \mu_\xi x_j(\xi) \geqslant \mu_{\xi_t} p_j^b(\xi_t)$$

for every non-terminal event ξ_t and every security $j \in J$. Therefore trading strategies permit a straightforward generalization of the fundamental theorem of asset pricing to markets with bid–ask spreads. When markets are frictionless, the payoff stream of an ask order is the opposite of the payoff stream of the bid order on this same security in the same event. Hence, the vector of event prices supports the market with an equality sign, and Theorem 3.1 coincides with the fundamental theorem of asset pricing (Harrison and Kreps (1979), Magill and Quinzii (1996)) expressed for trading strategies instead of portfolio strategies.

3.6 Characterization of Dynamic Completeness

Proposition 3.1 highlights the link between dynamic completeness and the positive span of payoff streams. Moreover, it allows us to use mathematical knowledge on positive spans to characterize dynamic completeness. To begin with, we show that a number of $\left\lceil \frac{k+1}{2(k+1-S)} \right\rceil$ traded securities is necessary for markets to be dynamically complete.

Proposition 3.2. Markets with bid–ask spreads are dynamically complete only if at least $\left\lceil \frac{k+1}{2(k+1-S)} \right\rceil$ securities are traded.

Proof. By definition, markets are dynamically completeness if $\mathcal{M}(p^a, p^b) = \mathbb{R}^k$. Then, p-span$(\phi) = \mathbb{R}^k$. Hence as a consequence of Corrolary 2.4 of Regis (2015) which states that any positive spanning set of $\mathbb{R}^k$ contains a basis of $\mathbb{R}^k$, the book order's payoffs matrix must have at least $k+1$ columns. It implies that at least $J \geqslant \frac{k+1}{2(k+1-S)}$ securities must be traded. Since J takes only integer values, markets are dynamically complete only if $J \geqslant \left\lceil \frac{k+1}{2(k+1-S)} \right\rceil$. □

Example 3.2. Consider a market with k future events and only one security $x \in \mathbb{R}^k$ available for trading. According to Proposition 3.2 markets with k events are dynamically complete only of at least $\frac{k+1}{2(k+1-S)}$ securities are traded. Hence, markets are dynamically complete only if the total number of events exceeds twice the number of states, that is $k+1 \geqslant 2S$.

The bound on the necessary number of traded security does not imply that a greater number of traded securities is necessary to satisfy dynamic completeness (see Section 2 for an example) in the presence of bid–ask spreads. However, the informativeness of this first result must be nuanced. Indeed, this bound is not informative whenever $S \leqslant \frac{k+1}{2}$ as it merely implies that at least one security must be traded. In any case, providing that this number of security is traded on the market is insufficient to ensure dynamic completeness.

It also depends on the values of the securities prices and dividends. In frictionless markets, dynamic completeness is equivalent to having a unique normalized strictly positive vector of supporting event prices under no-arbitrage. We show in the following theorem that the uniqueness of this vector is equivalent to having exclusively *non-zero period*-0 *event prices* supporting the market. A vector of event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ is said to have a non-zero period-0 price if $\mu_{\xi_0} \neq 0$.

Proposition 3.3. When markets are frictionless and admit no arbitrage opportunity, a unique normalized vector of strictly positive event prices supports the market if, and only if, every non-zero supporting event prices have a non-zero period-0 price.

Proof. First, we assume that there exists a unique normalized $\nu \in \mathbb{R}^{k+1}_{++}$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. We are going to show that every supporting event prices have a non-zero period-0 price. Assume by contradiction that there exist $\mu = \begin{pmatrix} 0 \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7 \setminus \{0\}$ with $\tilde{\mu} \in \mathbb{R}^k$ such that $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\epsilon > 0$ be such that $\nu + \epsilon\mu \in \mathbb{R}^{k+1}_{++}$, then $(\nu + \epsilon\mu)^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$ contradicting the uniqueness of ν.

Now, we assume that every $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^{k+1} \setminus \{0\}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^k$ satisfying $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$ are such that $\mu_{\xi_0} \neq 0$. We are going to show that there exists a unique normalized $\nu \in \mathbb{R}^{k+1}_{++}$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ with $\nu_{\xi_0} \in \mathbb{R}^*_+$ and $\tilde{\nu} \in \mathbb{R}^k_{++}$. Suppose by contradiction that there exists $\nu' = \begin{pmatrix} \nu'_{\xi_0} \\ \tilde{\nu}' \end{pmatrix}$ with $\nu'_{\xi_0} \in \mathbb{R}^*_+$ and $\nu_{\xi_0}{}' \in \mathbb{R}^k_{++}$ such that $\nu' \neq \lambda\nu$ for every $\lambda \in \mathbb{R}_+$ and $\nu'^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$. Let $\alpha \in \mathbb{R}$ be such that $\nu_{\xi_0} = \alpha\nu'_{\xi_0}$ and $(\nu - \alpha\nu')^\top z = 0$ implying $\nu = \alpha\nu'$, a contradiction.

□

We demonstrate in Theorem 3.2 that every supporting event prices vector must have a non-zero period-0 price.

Theorem 3.2. *The following propositions are equivalent:*

i. Markets are dynamically complete;

ii. for every non-zero payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ whose product with q is positive;

iii. every event price vector supporting the markets has a non-zero period-0 *price.*

Proof. The proof of i. equivalent to ii. follows from the characterization of positive spanning sets of Davis (1954). For the sake of clarity, we present it in our context. Assume that markets are dynamically complete. We are going to show that for every non-zero payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q\phi > 0$. Assume by contradiction that there exists a payoff stream $q \in \mathbb{R}^k$ such that $q\phi \leqslant 0$ for every $\phi \in \Phi$. Let $b \in \mathbb{R}_+^{2J(k+1-S)}$ be a trading strategy such that $z = \mathcal{P}b$, that is

$$z = \sum_{i=1}^{2J(k+1-S)} b_i \phi_i.$$

Moreover, $z^\top z > 0$ that is,

$$\sum_{i=1}^{2J(k+1-S)} b_i \phi_i^\top z > 0$$

implies that at least one element of the sum is positive. Hence there exists an order payoff stream ϕ_i such that $b_i \phi_i^\top z > 0$, a contradiction. Therefore, for every non-zero payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q\phi > 0$.

Now assume that for every payoff stream $q \in \mathbb{R}^k$ there exists an order payoff stream $\phi \in \Phi$ such that $q\phi > 0$. We are going to show that markets are dynamically complete. Assume by contradiction that markets are dynamically incomplete. It implies that $z \notin \text{p-span}(\Phi)$, that is $\text{p-span}(\Phi) \cap z = \emptyset$. Therefore according to Rockafellar (1970) Theorem 11.3, there exists a hyperplane with vector normal at the origin q' that properly separates $\text{p-span}(\phi)$ from the rest of $\mathbb{R}^k$, that is such that for either $q = q'$ or $q = -q'$, $q^\top \phi \leqslant 0$ for all $\phi \in \Phi$, a contradiction.

Now, we are going to show that ii. is equivalent to iii.. First, assume that for every payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. We are going to show that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^{k+1}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^k$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. Let $\nu_{\xi_0} \in \mathbb{R}$, $\tilde{\nu} \in \mathbb{R}^k$ and let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ be such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. By assumption, there exists $\phi \in \Phi$ such that $\phi^\top \tilde{\nu} > 0$. It implies $\nu_{\xi_0} \neq 0$.

Then, we assume that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^{k+1}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^k$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. We are going to show that for every payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. Assume by contradiction that there exists $\tilde{\nu} \in \mathbb{R}^k$, such that $\phi^\top \tilde{\nu} \leqslant 0$ for every $\phi \in \Phi$. Therefore, we have $\nu = \begin{pmatrix} 0 \\ \tilde{\nu} \end{pmatrix} \in \mathbb{R}^{k+1}$ such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, a contradiction.

□

In particular, Theorem 3.2 implies that it is necessary and sufficient to find a payoff stream with a non-positive product with every order payoff stream positive to demonstrate that markets are dynamically incomplete.

4 Conclusion

Market makers are essential actors of financial markets. To compensate for the risk they bear by holding securities for several periods, they charge positive transaction costs on trades creating frictions called bid–ask spreads. Bid–ask spreads represent the principal transaction costs borne by investors when trading on financial markets. We show that markets may be dynamically complete even in the presence of bid–ask spreads. Moreover, in some cases, removing bid–ask spreads will result in dynamically incomplete markets (see Example 2.1). Finally, we demonstrate that dynamic completeness is equivalent to the absence of frictionless supporting markets with a zero period-0 price.

Applications of our results concern the regulation of securities pricing in security exchanges, particularly the size of acceptable bid–ask spreads. Competition between market makers prevents bid–ask spreads from being excessively large; nonetheless, the question remains whether a complementary regulation is necessary to achieve specific goals such as completeness (see Duffie and Rahi (1995)). Another potential application of these results concerns replacing post-trade intermediaries on security exchanges with distributed ledgers technologies (DLTs). These technologies provide the possibility of disposing of intermediaries in trades and are expected to reduce transaction costs. Glosten (1994) shows that DLTs provide as much liquidity as can be expected in extreme adverse selection environments. Nonetheless, he nuances the expectation that DLTs necessarily cut transaction costs by demonstrating that the spread in small trades in electronic limit-order is positive. In contrast, it is possible to imagine a competitive pricing model with zero small-trade spread (see Glosten (1989)). DLTs' proponents must demonstrate that they outclass the current market organization to gain financial regulators' support. One of the various questions they should answer is: regarding risk-sharing, will securities exchanges benefit from switching to DLTs?

Equally, recent literature on asset pricing in markets with frictions provides closed-form expression to pricing rules. However, a trade-off for the increased precision is that some authors assume markets are complete. It is the case in particular, in Cerreia-Vioglio, Maccheroni and Marinacci (2015) and Araujo, Chateauneuf and Faro (2012) which are two of the most significant models of the field. Indeed, Araujo, Chateauneuf and Faro (2012) assume the pricing rule is the super-replication price of some underlying incomplete security market. It amounts to assume completeness in the traditional sense that it is possible to trade every payoff stream. It is difficult to improve these models by extending them to multiperiod settings as preliminary theoretical questions must be addressed. For example, Araujo, Chateauneuf, Faro and Holanda (2019) determine which properties of Araujo, Chateauneuf and Faro (2012) and Cerreia-Vioglio, Maccheroni and Marinacci (2015) are stable in 3−period security markets. Naturally, another question relates to the assumption that markets are complete. In 2−period security markets, completeness depends exclusively on the rank of the dividend matrix. It is unaffected by the presence of friction. In multiperiod security markets, prices impact the available payoff streams through trading happening at intermediary periods. Therefore non-linearities modify the characterization of dynamic completeness. Hence, any attempt to extend these results to multiperiod settings necessitates characterizing dynamic completeness and determining how compelling it is in the presence of friction. Araujo, Chateauneuf, Faro and Holanda (2019) do not address this question. They adopt a non-standard definition of completeness: the payoffs received at the intermediary period are not part of it. It is as if agents initially ignore that markets re-open at the intermediary period.

A significant improvement to our contribution is to characterize dynamic completeness when it is explicit that the traded quantity impacts the unitary price. Indeed, in security markets, market makers provide ask and bid offers for specific quantities. Therefore, prices are not linear in quantity purchased or sold. For example, in Kyle (1985), Glosten (1994) and Biais, Martimort and Rochet (2000), prices are convex and increasing in the traded quantity. This will be the subject of future research.

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