# Root Dominance - Job Market Paper

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**Abstract**

Root dominance is an intermediate dominance relation between weak and strict dominances. In addition to weak dominance, root dominance requires strict dominance on all profiles where an opponent plays a best response to the dominating strategy. The iterated elimination of root dominated strategies (IERDS) outcome refines the iterated elimination of strictly dominated strategies (IESDS) outcome, and IERDS is an order independent procedure in finite games, contrary to the iterated elimination of weakly dominated strategies (IEWDS). In addition, IERDS does not face the inconsistency that we call mutability. That is, IERDS does not alter the dominance relation between two strategies like IEWDS does. Finally, we introduce a rationality concept which corresponds to root undominated strategies. This rationality concept is induced by perturbations of the game such that a player believes the strategies he is considering might be observable by his opponent. We discuss the links between our concept and other concepts established in various literatures such as the conjectural variations theory.

*Keywords:* Dominance relations; Iterated elimination procedures; Rationality

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# 1 Introduction

## 1.1 Motivating example

Assume two agents who have coordination incentives but also have strong egocentric biases. That is, each agent is indifferent between, on the one hand, coordinating on his least preferred action with the other agent and, on the other hand, miscoordinating but choosing his preferred action. This situation can be represented in the following game which can be seen as a modified version of the battle of the sexes (BoS) where best responses payoffs are underlined[1]:

| | $j$'s Strategy | | |
|---|---|---|---|
| $i$'s Strategy | $A_j$ | $B_j$ | $O_j$ |
| $A_i$ | $(\underline{3},\underline{2})$ | $(\underline{2},\underline{2})$ | $(\underline{1},0)$ |
| $B_i$ | $(1,1)$ | $(\underline{2},\underline{3})$ | $(0,0)$ |
| $O_i$ | $(0,0)$ | $(0,\underline{1})$ | $(\underline{1},\underline{1})$ |

Figure 1: Modified Version of the Battle of the Sexes

Remark first that no strategy is strictly dominated. Thus, the iterated elimination of strictly dominated strategies (IESDS) does not eliminate any strategy. In contrast, both outside options $O_i$ and $O_j$ are weakly dominated (respectively by $A_i$ and $B_j$). As well, $B_i$ and $A_j$ are weakly dominated[2]. However, as noted by Samuelson (1992):

> It is well known that the order in which dominated strategies are eliminated can affect the outcome of the [iterated elimination of weakly dominated strategies (IEWDS)].

In other words, IEWDS is order dependent (see also Marx and Swinkels (1997); Hillas and Samet (2020)). Here, it is the case since IEWDS always eliminates outside options $O_i$ and $O_j$ but only sometimes $A_j$ and/or $B_i$. It is striking that no iterated elimination procedure based on a dominance relation[3] can both provide a unique outcome when applied to this game and still eliminate some strategies. Particularly, it

[1]Remark that utility functions can be denoted: $U_i(A_i) = 2 + \mathbb{1}_{A_j} - \mathbb{1}_{O_j}$, $U_i(B_i) = 1 + \mathbb{1}_{B_j} - \mathbb{1}_{O_j}$, $U_i(O_i) = 0 + \mathbb{1}_{O_j}$; and in a symmetric way for player $j$: $U_j(B_j) = 2 + \mathbb{1}_{B_i} - \mathbb{1}_{O_i}$, $U_j(A_j) = 1 + \mathbb{1}_{A_i} - \mathbb{1}_{O_i}$, $U_j(O_j) = 0 + \mathbb{1}_{O_i}$.

[2]In addition to the ouside options, this is the main difference with the standard BoS.

[3]See Definition 1 for the precise definition.

is remarkable that even the Nash equilibrium $(O_i, O_j)$ cannot be ruled out while we could intuitively think that players "should" try to coordinate on better outcomes. In this paper, we introduce a new dominance relation named *root dominance* and an associated order independent iterated elimination procedure the *iterated elimination of root dominated strategies (IERDS)* such that IERDS eliminates both $O_i$ and $O_j$ and stops there. Root dominance requires weak dominance and strict dominance on *all* the profiles where the opponent best responds to the dominating strategy. In our version of the Battle of the Sexes, $j$ best responds to $A_i$ by playing $A_j$ or $B_j$. At these two profiles, $A_i$ strictly payoff dominates $O_i$. Therefore, $A_i$ root dominates $O_i$. On the contrary, playing $A_i$ does not yield a strictly higher payoff than playing $B_i$ when $j$ plays $B_j$. Thus, $A_i$ does not root dominate $B_i$ and $B_i$ is never eliminated by IERDS.

### 1.2 Elimination procedures based on dominance relations

Iterated elimination of strictly dominated strategies (IESDS) is one of the most basic tools of game theory. It is among the least vulnerable solution concepts when analysts eliminate strategies to predict the outcome of a situation. Notably, it is equivalent to the concept of rationalizability in two-player games (see Bernheim (1984); Pearce (1984)) and when a game is dominance solvable[4], it reinforces the use of the Nash equilibrium as a solution concept, like in the Cournot duopoly. Remarkably, for instance, IESDS is essential to understand why there is a unique equilibrium in global games (see Carlsson and van Damme (1993)). However, the conceptual robustness of IESDS necessarily reduces its use when precise predictions are required. Instead, iterated elimination of weakly dominated strategies (IEWDS) outcome is a refinement of IESDS outcome. IEWDS has been largely applied in different strands of the economic literature such that the voting literature (see Moulin (1979)). Additionally, a certain order of IEWDS is equivalent to the backward induction solution[5] (see Moulin (1986, p.84)). Though, IEWDS may go sometimes "too far" in the selection. As an example, it may eliminate the only Nash equilibrium in certain games such that the Bertrand duopoly. Furthermore, inconsistencies of IEWDS refrain its use as a solution concept. In particular, order dependence[6] of IEWDS (and therefore the multiplicity of final outcomes) prevents firm forecasts. However, attempts to justify the use of IEWDS have been made. Among this literature, Marx and Swinkels (1997) shows that IEWDS is

[4]Dominance solvability means that IESDS outcome is a unique profile.

[5]It is true in games where, if a player is indifferent between two terminal nodes, it implies that all players are indifferent at these same terminal nodes. Moulin (1986)) calls this assumption the one-to-one assumption.

[6]It means that different applications of the procedure may lead to different final outcomes. See Section 2 for definitions. The problem of order independence of procedures has given a rich literature (see for instance Gilboa et al. (1990); Apt (2005, 2011); Luo et al. (2020); Hillas and Samet (2020)).

*payoffs* order independent in games with *transference of decisionmaker indifference (TDI)*[7], and define in association, the *nice weak domination*[8]. Nevertheless, the order independence result is limited to payoffs (and does not apply to strategies)[9], while in the context of decision theory, Kahneman and Tversky (1979) show that payoffs may not determine entirely the preferences. Then, from both theoretical and practical points of view, payoffs independence might not be considered as strong a result as strategies order independence. Alternatively, we propose in this paper a dominance relation and an associated procedure whose outcome refines the IESDS outcome and is (payoff and strategies) order independent in every finite game.

### 1.3 Outline

We introduce in this paper a new kind of iterated elimination procedure based on a new dominance relation called *root dominance*. Root dominance is a stronger relation than weak dominance and weaker than strict dominance. That is, root dominance requires weak dominance and the strict payoff dominance on a specific profile set: the *best reply set* to the dominating strategy. Note that this last property depends essentially on the dominating strategy, which is, to the best of our knowledge, a novelty. We introduce also a new iterated elimination procedure, whose order independence property is not limited to payoffs, but concerns strategies as well.

In the next section, we establish a simplified framework with only pure strategies. In Section 3, we define the notion of root dominance and our iterated elimination procedure IERDS. Additionally, we illustrate them with some examples. In Section 4, we show the technical lemmas and the order independence result. We make a succinct literature review about iterated elimination procedures in Section 5. Then, in Section 6, we present the mutability issue, notably faced by IEWDS, and show that IERDS is immutable. In Section 7, we extend our concepts to a framework with mixed strategies and show that our results hold true. We introduce our rationality concepts in Section 8 and we compare them specifically to the concepts in conjectural variation theory concepts. Finally, we conclude in Section 9.

[7] A game exhibits *TDI* when, if one agent is indifferent between two strategies at a given opponents' profile, every player is indifferent between the two profiles formed by either one or the other strategy of the first player, and the given opponents' profile.

[8] A strategy $s_i'$ of player $i$ is said *nicely weakly dominated* by strategy $s_i''$ if, in addition to weak dominance, everywhere where $i$ is indifferent between $s_i'$ and $s_i''$, $i$'s opponents are also indifferent between $i$ playing $s_i'$ and $s_i''$.

[9] See Appendix D to distinguish our notion of order independence and the Marx and Swinkels (1997)'s one.

## 2 Framework with pure strategies

We denote $\Gamma = \{I, S, U\}$ a finite game with $I$ the set of players, $S = \prod_{i \in I} S_i$, $S_i$ being the finite strategy set of player $i \in I$ (we consider only pure strategies), and $U$ the vector of utility functions of each player $i$ where $U_i : S \to \mathbb{R}$. We denote $S_{-i} = \prod_{j \in I \setminus \{i\}} S_j$ the strategy profiles set of $i$'s opponents. Finally, we denote $s \in S$ a strategy profile, and $s_{-i} \in S_{-i}$ the strategy profile of the opponents of $i \in I$ such that when $i$ plays $s_i$, $s = (s_i, s_{-i})$.

Here, we define the main notion that motivates this paper, namely *order independence*. Before, we define a process associated with any dominance relation: A *process* iteratively eliminates *some* dominated strategies *at the step they are eliminated* with *a specific order*, and ends when there is no dominated strategy anymore. Then, a *procedure* associated with a game is the class of all *processes* applied to the game.

Now, we can state what we mean by order independence when we study a precise game:

**Definition 1.** *A procedure associated with a dominance relation and a game is said order independent for this game if all processes have the same final (strategies) outcome.*

Importantly, the final outcome of a process contains the payoffs and the strategies. Again, this feature distinguishes ourselves from Marx and Swinkels (1997) who look only at payoffs to define order independence[10]. Finally, we define *order independence* for the class of games we study, namely the finite games:

**Definition 2.** *A procedure associated with a dominance relation is said order independent if it is order independent for every finite game.*

In the next section, we define formally the *sequence of games*[11] associated with a process, which further specifies the kind of order independence we consider. Importantly, except explicit mention, we consider that a procedure is order independent if and only if any number (but zero) of strategies can be eliminated at each step of the processes run by the procedure and all processes have the same final outcome.

[10]Obviously, both notions often coincide, but it is not always the case.
[11]See Definition 5 below for the formal definition.

# 3 Root dominance

In this section, we define our dominance relation as well as our iterated elimination procedure.

## 3.1 The dominance relation

To establish the dominance relations in this section, we first redefine a standard notion of game theory, the *Best Reply Set to a strategy*:

**Definition 3.** *The Best Reply Set to $s_i'' \in S_i$, denoted $\boldsymbol{b}(s_i'')$, is the set of all strategy profiles $s^* \in S$ such that:*
*$s_i^* = s_i''$, and, if $S_{-i} \neq \emptyset$:*

$$\exists j \in I \setminus \{i\},\ \ s_j^* \in \arg\max_{s_j \in S_j} U_j(s_j, s_{-j}^*) \tag{OM}$$

The Best Reply Set is simply the set of all profiles which contain $s_i''$ and where at least one $i$'s opponent best responds to the profile (OM′). If there is no opponent or their strategy sets are empty, the Best Reply Set is simply the strategy $s_i''$.

Now, we define our dominance relation, namely *root dominance*:

**Definition 4.** *A strategy $s_i' \in S_i$ is said root dominated by the strategy $s_i'' \in S_i$, (denoted $s_i'' \succ s_i'$), if:*

$$\forall s_{-i} \in S_{-i} : \qquad U_i(s_i'', s_{-i}) \geq U_i(s_i', s_{-i}) \tag{RD1}$$

$$\forall s_{-i}^* \text{ such that } s^* \in \boldsymbol{b}(s_i'') : U_i(s_i'', s_{-i}^*) > U_i(s_i', s_{-i}^*) \tag{RD2}$$

RD1 and RD2 are *inadmissibility* conditions, i.e., they ensure that root dominated strategies are weakly dominated. Precisely, RD1 states that $s_i'$ is *very weakly dominated* by $s_i''$. There is *very weak dominance* if a strategy always pays off at least as much as another strategy (see Marx and Swinkels (1997) for a formal definition). Therefore, either the former strategy (weakly) dominates the latter, or they are equivalent. RD2 states that $s_i''$ is strictly preferred to $s_i'$ if the opponents play a profile in $\boldsymbol{b}(s_i'')$. Additionally, we will denote respectively the strict and the weak dominance relation:

$$s_i'' \underset{\text{S}}{\succ} s_i' \text{ and } s_i'' \underset{\text{W}}{\succ} s_i'$$

## 3.2 Finite sequence of games

Since we are interested in defining iterated elimination procedures and comparing them to IEWDS and IESDS, we formally define the *sequence of games* that will be used in this section, in association with the dominance relation we have defined above:

**Definition 5.** *A sequence of games associated with a game* $\Gamma$ *is:*

$$\{\Gamma^\lambda\}_{\lambda\leq\Lambda} \equiv \{\Gamma^0 \equiv \Gamma, \dots, \Gamma^\lambda, \dots, \Gamma^\Lambda\}$$

*with* $\lambda \in [\![0, \Lambda]\!]$ *such that:*

- $\forall \lambda \in [\![0, \Lambda]\!],\ \Gamma^\lambda = \{I, S^\lambda, U\}$, *with* $S^\lambda = \prod_{i\in I} S_i^\lambda$, $S_i^\lambda$ *being the strategy set of player* $i \in I$, $I$ *the unchanged set of players of* $\Gamma$, *and* $U$ *the vector of utility functions of each player* $i$ *(whose domain is restricted),* $U_i : S^\lambda \to \mathbb{R}$,
- $\forall \lambda \in [\![1, \Lambda]\!],\ \Gamma^\lambda$ *is a restriction of* $\Gamma^{\lambda-1}$, *i.e.,* $S^\lambda = \prod_{i\in I} S_i^{\lambda-1} \setminus \underline{S_i^{\lambda-1}}$ *where for each player* $i$, $\underline{S_i^{\lambda-1}}$ *is an arbitrary (possibly empty) set of strategies in* $S_i^{\lambda-1}$ *dominated in* $\Gamma^{\lambda-1}$, *but such that for at least one player* $i \in I$, $\underline{S_i^{\lambda-1}}$ *is non empty.*
- $\underline{S^\lambda} = \prod_{i\in I} \emptyset$ *if and only if* $\lambda = \Lambda$.

The sequence of games starts from the original game $\Gamma$, and then restricts the strategy set by eliminating *some* (i.e. at least one but not necessarily all) dominated strategies at each step of the sequence. The sequence ends if and only if no more strategy is dominated. Then, we can define the *iterated elimination of root dominated strategies (IERDS)* as the procedure that iteratively eliminates *some* root dominated strategies *at the step they are eliminated* and ends when there is no root dominated strategy anymore. As explained above, the procedure can lead to several processes, each one associated to a sequence of games.

Let us study how root dominance and IERDS work in finite games through the next example:

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(\underline{4},\underline{2})$ | (1,1) |
| $B$ | $(2,\underline{2})$ | $(\underline{4},\underline{2})$ |
| $O$ | $(2,\underline{2})$ | $(2,\underline{2})$ |

$\xrightarrow{\text{IERDS}}$

| | $j$'s S. |
|---|---|
| $i$'s S. | $L$ |
| $T$ | $(\underline{4},\underline{2})$ |

Figure 2: Game with a Unique Prediction

$O$ is not root dominated by $B$ nor by $T$. Even if $O$ is (very) weakly dominated by $B$ (RD1 is thus respected), we see that both $(B, L)$ and $(B, R)$ are in $\boldsymbol{b}(B)$, and since $U_i(B, L) = U_i(Z, L)$, there is no root dominance since it requires strict dominance on all profiles in $\boldsymbol{b}(B)$ (RD2). $T$ does not either, because of RD1. Indeed, there is no (very) weak dominance since $U_i(T, R) < U_i(Z, R)$. However RD2 is checked since $\boldsymbol{b}(T) = (T, L)$ and $U_i(T, L) > U_i(Z, L)$. Concerning player $j$, $L$ root dominates $R$. Actually, $L$ (very) weakly dominates $R$ and $U_j(T, L) > U_j(T, R)$ while $\boldsymbol{b}(L) = (T, L)$. After eliminating $R$, we see that both $B$ and $O$ are root dominated since $T$ strictly dominates them. Finally, IERDS selects $(T, L)$ like IEWDS.

Before focusing ourselves on the results, we make a semantical precision: we say that $s'_i$ is *eliminated* by $s''_i$ at step $\lambda$ of a sequence of games if:

$$s''_i \succ s'_i, \text{ and } s''_i \in S_i^{\lambda+1}, \text{ and } \; s'_i \in S_i^{\lambda} \setminus S_i^{\lambda+1}.$$

Obviously, $s'_i$ is *eliminated* by $s''_i$ only if it is root dominated by $s''_i$, but the converse is not necessarily true in a given process. The reason is that both $s'_i$ and $s''_i$, or only $s''_i$ or neither of them might be eliminated at a given step. However, for the case of root dominance and IERDS, the distinction between domination and elimination is only made to ease the establishment of the next results. That is, a root dominated strategy always has an undominated dominator in finite games, and then, for each root dominated strategy, one can find a strategy that *eliminates* it. We formally prove this statement below in Lemma 2.

# 4 Order independence result

## 4.1 Technical results

**Lemma 1.** $\forall i \in I, \forall s_i \in S_i, \boldsymbol{b}(s_i) \neq \emptyset$

*Proof.* By Definition 3, it is straightforward that $\boldsymbol{b}(s_i)$ is never empty for any finite game. Indeed, either there is no opponent (or equivalently opponents' strategy sets are empty) and then $\boldsymbol{b}(s_i) = s_i$. Otherwise, since the game is finite, each player has (at least) a best response to each strategy profiles of his opponents. ■

Now, we state that root dominance forms a strict partial order:

**Proposition 1.** *With respect to a fixed game, root dominance induces a strict partial order on the strategy set of any player $i \in I$: it is a binary relation such that irreflexivity, asymmetry and transitivity hold.*

*Proof.* Root dominance is irreflexive: by Lemma 1, $\boldsymbol{b}(s_i'') \neq \emptyset$, and it is not possible to have $U_i(s_i, s_{-i}) > U_i(s_i, s_{-i})$ for any profile $s_{-i} \in S_{-i}$. Then, RD2 cannot be respected. Root dominance is transitive: assume $s_i'' \succ s_i'$ and $s_i''' \succ s_i''$. Here, we have to prove that $s_i''' \succ s_i'$. First, it is straightforward that RD1 is respected. Second, since $s_i''' \succ s_i''$, we know that $U_i(s_i''', s_{-i}) > U_i(s_i'', s_{-i})$ for each strategy profile $s_{-i}$ contained in $\boldsymbol{b}(s_i''')$. Since $s_i'' \succ s_i'$, $U_i(s_i'', s_{-i}) \geq U_i(s_i', s_{-i})$ for each strategy profile $s_{-i}$ in $S_{-i}$, and thus for each strategy profile $s_{-i}$ contained in $\boldsymbol{b}(s_i''')$. Therefore, $U_i(s_i''', s_{-i}) > U_i(s_i'', s_{-i}) \geq U_i(s_i', s_{-i})$ for each strategy profile $s_{-i}$ contained in $\boldsymbol{b}(s_i''')$ and RD2 is respected. Finally, irreflexivity and transitivity together imply asymmetry. ■

**Lemma 2.** *If $s_i' \in S_i$ is root dominated, there is (at least) one strategy $s_i'' \in S_i$ that may eliminate it, i.e., $s_i''$ is not root dominated by any strategy in $S_i$ and $s_i''$ root dominates $s_i'$.*

*Proof.* Since the number of strategies is finite, the number of strategies root dominating $s_i'$ is necessarily finite. Let us denote it $m$ and denote $g(s_i')$ the set of these strategies. Then, (at most) $m-1$ of these strategies are root dominated. Otherwise, it means that the $m^{th}$ strategy is root dominated by an other strategy outside $g(s_i')$[12]. By transitivity of root dominance, it means that the latter strategy also root dominates $s_i'$, contradicting the fact that the number of strategies root dominating $s_i'$ is $m$. If less than $m-1$ strategies are root dominated, we do have that there is (at least) one strategy that is not root dominated by an other strategy and which root dominates $s_i'$. ■

The next lemma establishes that the set $\boldsymbol{b}(s_i)$ never expands as we progress through the steps of IERDS:

**Lemma 3.** $\forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda},\ \forall \lambda \in [\![0, \Lambda - 1]\!],,\ \forall i \in I,\ \forall s_i \in S_i^{\lambda+1},$

$$\boldsymbol{b}^{\lambda+1}(s_i) \subseteq \boldsymbol{b}^{\lambda}(s_i).$$

*Proof.* Assume there exists a profile $s' \equiv \prod_{k \in I} s_k' \in \boldsymbol{b}^{\lambda+1}(s_i) \setminus \boldsymbol{b}^{\lambda}(s_i)$. Since $s' \notin \boldsymbol{b}^{\lambda}(s_i)$ but $s' \in \boldsymbol{b}^{\lambda+1}(s_i)$, we know that there is no best response in $s'$ at $\lambda$ but also that (at least) one player $j$ best responds with the strategy $s_j'$ to $s_{-j}'$ at $\lambda+1$. Thus, we assume that there is (at least) one player $j \neq i$ with a best response $s_j'' \in S_j^{\lambda}$ to $s_{-j}'$, eliminated at step $\lambda + 1$ such that:

$$U_j(s_j'', s_{-j}') > U_j(s_j', s_{-j}').$$

[12] By Proposition 1, root dominance is asymmetric and transitive. Then, there is at least one strategy (the $m^{th}$ here) that is not root dominated by a strategy in $g(s_i')$. Indeed, if each strategy is root dominated by a strategy in $g(s_i')$, one can find a contradiction with asymmetry and transitivity.

Since $s''_j$ is root dominated, then by Lemma 2 $s''_j$ is root dominated by (at least) an uneliminated strategy $s'''_j$, present at step $\lambda+1$. Since $s''_j$ is a best response for $j$ to the profile $s'_{-j}$, we necessarily have $U_j(s''_j, s'_{-j}) = U_j(s'''_j, s'_{-j}) > U_j(s'_j, s'_{-j})$. Therefore, at step $\lambda+1$, player $j$ still wants to deviate from $s'_j$ to $s'''_j$. It contradicts the hypothesis that $s'_j$ is a best response for $j$ at step $\lambda+1$ and finally it contradicts that $s' \subseteq \boldsymbol{b}^{\lambda+1}(s_i)$.

■

This property would not be true if, for instance, we considered only profiles where each opponent plays a best response. Clearly, either these profiles could not exist, or they could be eliminated (see Appendix C), inducing new profiles in $\boldsymbol{b}(s_i)$ where a "new" maximal payoff would be obtained.

Now we establish that the relation of root dominance between two strategies is maintained through the steps of IERDS:

**Lemma 4.** *$\forall \{\Gamma^\lambda\}_{\lambda\leq\Lambda}$, $\forall \lambda \in [\![0, \Lambda-2]\!]$, $\forall i \in I$, $\forall s'_i, s''_i \in S_i^{\lambda+1}$, if $s''_i \succ s'_i$ in $\Gamma^\lambda$, then $s''_i \succ s'_i$ in $\Gamma^{\lambda+1}$.*

*Proof.* Assume $s''_i \succ s'_i$ in $\Gamma^\lambda$. It is straightforward that RD1 is still verified in $\Gamma^{\lambda+1}$. By Lemma 3, we know that for any strategy $s_i$, $\boldsymbol{b}^{\lambda+1}(s_i) \subseteq \boldsymbol{b}^\lambda(s_i)$. Therefore RD2, is still verified as well. ■

Note that $\boldsymbol{b}^\lambda(s''_i)$ being not empty for each $\lambda$ by Lemma 1, there is still a profile such $s''_i$ strictly payoff dominates $s'_i$. Besides, remark that we consider only $\lambda \in [\![0, \Lambda-2]\!]$ for a given sequence because in $\Gamma^\Lambda$ no strategy is root dominated.

Now, we define a notion introduced by Apt (2011), namely the *hereditariness* of a dominance relation. *Hereditariness* is useful to establish order independence of the procedure associated with the dominance relation which verifies it. Denote $c(\Gamma)$, the $\Gamma$-choice, i.e. the set of strategies in $S$ which are not dominated in $\Gamma$ (given a dominance relation). *Hereditariness* means that no strategy previously dominated becomes non dominated after one step of a process:

**Definition 6.** *A dominance relation is said to verify hereditariness if* $\forall \{\Gamma^\lambda\}_{\lambda\leq\Lambda}, \forall \lambda \in [\![0, \Lambda-1]\!]$,

$$\Gamma^\lambda, \Gamma^{\lambda+1} \in \{\Gamma^\lambda\}_{\lambda\leq\Lambda} \Rightarrow c(\Gamma^{\lambda+1}) \subseteq c(\Gamma^\lambda).$$

Note that *hereditariness* is called *1-Monotonicity** in Luo et al. (2020). Here, we verify that root dominance is hereditary:

**Lemma 5.** *Root dominance verifies hereditariness. It is also equivalent to the following statement:* $\forall\{\Gamma^\lambda\}_{\lambda\leq\Lambda}$, $\forall\lambda\in[\![0,\Lambda-2]\!]$, $\forall i\in I$, $\forall s'_i\in S_i^{\lambda+1}$, *if* $s''_i\in S_i^\lambda$ *root dominates* $s'_i$ *in* $\Gamma^\lambda$, *then* $s'_i$ *is still root dominated in* $\Gamma^{\lambda+1}$.

*Proof.* First, if $s''_i\in S_i^{\lambda+1}$, by Lemma 4, the result is immediate, i.e. RD1 and RD2 are still respected. Second, if $s''_i\notin S_i^{\lambda+1}$, then by Lemma 2 there is (at least) a strategy $s'''_i$ that eliminates $s''_i$. By Proposition 1, each strategy that root dominates $s''_i$ root dominates $s'_i$ as well in $\Gamma^\lambda$. Thus, there is still (at least) one strategy that root dominates $s'_i$ in $\Gamma^{\lambda+1}$. ■

By Apt (2011, Theorem 1), we know that hereditariness implies, in finite games, order independence of the procedure associated with the dominance relation. The following result, a corollary of Lemma 5, illustrates with another perspective why our elimination procedure is order independent. In the words of Dufwenberg and Stegeman (2002); Luo et al. (2020), each root dominated strategy has an undominated root dominator, i.e. each root dominated strategy at a point of a sequence of games will be deleted by the end of the sequence:

**Proposition 2.** $\forall\{\Gamma^\lambda\}_{\lambda\leq\Lambda}$, $\forall\lambda\in[\![0,\Lambda-1]\!]$, $\forall i\in I$, $\forall s_i\in S_i^\lambda$ *root dominated in* $\Gamma^\lambda$ ,

$$s_i\notin\Gamma^\Lambda.$$

*Proof.* The proof is made by applying an induction reasoning on Lemma 5. Assume a process of IERDS applied to the game $\Gamma$, and the associated sequence of games $\{\Gamma^\lambda\}_{\lambda\leq\Lambda}$. Assume $s_i$ is root dominated at step $\lambda-1$. By the definition of the sequences of games, $s_i$ is eliminated, and $s_i$ is not in $\Gamma^\Lambda$. Now assume the property that *a root dominated strategy* $s_i$ *at* $\Lambda-\mu$ *is not in* $\Gamma^\Lambda$ *for a given* $\mu\in[\![2,\Lambda-1]\!]$ is true. Let us show it is true for $\mu+1$. Thus, assume that $s_i$ is root dominated at $\Lambda-(\mu+1)$. Either $s_i$ is eliminated at this step and we have the result, or, it is not eliminated. In this latter case, by Lemma 5, $s_i$ is root dominated at $\Lambda-\mu$, and therefore, we have the result by the induction hypothesis. We have shown that a strategy root dominated at $\Lambda-(\mu+1)$ was deleted by the end of the sequence. Thus, by induction, it is true for each $\mu\in[\![2,\Lambda]\!]$. Since we did not need any assumption on the process used to construct our initial sequence, this result is true for any process. ■

## 4.2 Main result

**Theorem 1.** *IERDS is order independent in finite games.*

*Proof.* By Lemma 5 and Apt (2011, Theorem 1), the result is immediate. ■

# 5 Related literature about other elimination procedures

In an unifying framework gathering weak and strict dominances, Hillas and Samet (2020) eliminate *flaws*, i.e., strategy profiles rather than strategies. A flaw deletion occurs if playing the given flaw implies that an agent plays a dominated strategy. If flaws elimination is used, then weak and strict dominance are order independent in finite games (Hillas and Samet (2020, Proposition 1)). Therefore, weak dominance rationality seems to be as legitimate as strict dominance rationality if iterated elimination of flaws is considered[13]. Nevertheless, the purpose in Hillas and Samet (2020) is mainly to rationalize the use of weak dominance. Moreover, the iterative elimination of flaws does not actually eliminate the profiles or strategies from the original game that is considered. Rather, eliminated profiles or strategies are seen as not playable by the agents, but they may be used in order to justify further flaws deletions.

In the same vein, Asheim and Dufwenberg (2003) refine the notion of *permissibility* of Dekel and Fudenberg (1990)[14] with *full permissibility sets* and the associated *iterated elimination of choice sets under full admissible consistency* (IECFA). IECFA considers strategy subsets (and not strategies like in IESDS or IEWDS). Roughly speaking, IECFA eliminates weakly dominated strategies, and then keeps a strategy subset of the first player if there is at least a surviving opponent's subset such that considering only the profiles contained in this opponent's subset, the strategies in the subset of the first player are the only undominated strategies (i.e. not weakly dominated strategies). The outcome of IECFA is made of subsets. All these subsets can correspond to a belief about a surviving opponent's subset, but the beliefs do not have to be consistent between players (like in rationalizability and contrary to Nash equilibrium). IECFA is order independent by definition. Indeed, like the Dekel-Fudenberg (DF) procedure of Dekel and Fudenberg (1990), each eliminable strategy (subset) is eliminated at each step. Nevertheless, the outcome still exhibits multiplicity.

An other procedure based on beliefs is the *reasoning-based expected utility procedure* (RBEU) of Cubitt and Sugden (2011). RBEU is an iterated procedure in which strategies are *accumulated*[15] if there is no belief such that another strategy gives a strictly higher payoff to the player (the strategy is dominant). If a player's strategy

[13]Interestingly, in finite games, the outcome of IEWDS is contained in the outcome of weak flaws elimination and the outcome of IESDS is equal to the outcome of strict flaws elimination. Then, one may wonder whether the order independence of IESDS in finite games may only be due to the fact that IESDS is incidentally equivalent to strict flaws elimination.

[14]A strategy is permissible if, after one round of elimination of all the weakly dominated strategies, it survives to the iterated elimination of strictly dominated strategies.

[15]Briefly, accumulated strategies are the undeletable strategies.

is accumulated, the procedure allows only opponents' beliefs which allocate a strictly positive probability to the occurrence of this strategy. Strategies which are always strictly dominated for these beliefs are deleted and so on. It is immediate by its definition that RBEU deletes (at least) as many strategies as IESDS in finite games. Thus, RBEU refines IESDS. Moreover, it is order independent in finite games, contrary to IEWDS. However, RBEU refines strictly IESDS if and only if there is (at least) a dominant strategy, a quite huge requirement.

# 6 The mutability problem

Now, we define the second consistency requirement we are concerned with, namely *immutabilty*. Note that we call *mutability* what Cubitt and Sugden (2011) call "undercutting problems" and what Hillas and Samet (2020) call "inconsistency". Samuelson (1992) contrasts iterated admissibility (i.e. IEWDS) and common knowledge of admissibility by emphasizing this inconsistency with the following words:

> The difference in these two outcomes reflects the fact that once a strategy [...] is eliminated by iterated admissibility, it cannot return even if the reason for its elimination has been purged.

First, we introduce the notion of *virtual* domination:

**Definition 7.** *A strategy eliminated by a process is said virtually dominated if, added to the final outcome of the process, it is a dominated strategy.*

**Definition 8.** *A procedure is immutable (for a given game) if in each process associated to it (for this given game), all eliminated strategies are virtually dominated.*

Table 1 summarizes the inconsistencies associated to the procedures we have mentioned above. Now, we study in details these "inconsistencies" of IEWDS through various versions of an example taken in Hillas and Samet (2020). Note that such remarks had been already formulated in Samuelson (1992) for instance. We compare IERDS to the solution of Hillas and Samet (2020) to deal with these inconsistency problems, namely the *flaws elimination* or also called *deletion of inferior profiles*[16]. Following Stalnaker (1994), Hillas and Samet (2020) propose to eliminate profiles (rather than strategies) such that if they were played, it would mean that a (weakly)

[16]See also Bonanno and Tsakas (2018) who study the properties of the so-called *iterated deletion of inferior profiles (IDIP)* in a framework with ordinal utilities.

| Inconsistencies | Definitions | Procedures |
|---|---|---|
| Order Dependence | The order of elimination affects the final outcome | IEWDS |
| Mutability | A strategy may be virtually not dominated whereas it is dominated at a previous step | IEWDS, IECFA, DF |

Table 1: Inconsistencies of Elimination Procedures in Finite Games

dominated strategy is effectively played. We illustrate mutability with the following example, such that $\underset{\text{IEWDS}}{\longrightarrow}$ means that a process of IEWDS is run (and gives the final outcome when cells color is blank), and the cells in blue indicates eliminated strategies but which are non virtually dominated after the process has been terminated:

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(\underline{2},\underline{1})$ | $(\underline{3},0)$ |
| $B$ | $(\underline{2},0)$ | $(2,\underline{1})$ |

$\underset{\text{IEWDS}}{\longrightarrow}$

| | $j$'s Strat. |
|---|---|
| $i$'s Strat. | $L$ |
| $T$ | $(\underline{2},\underline{1})$ |
| $B$ | $(\underline{2},0)$ |

Figure 3: Hillas and Samet (2020)'s Game with IEWDS Mutability

The game of Figure 3 has one pure Nash equilibrium $(T, L)$. $T$ weakly dominates $B$. If $B$ is eliminated, then $R$ is strictly dominated and the surviving outcome is $(T, L)$, the pure Nash equilibrium. However, as mentioned by Hillas and Samet (2020), this iterated deletion is inconsistent. Indeed, if $R$ is eliminated, then $B$ is not weakly dominated anymore and then $(T, L)$ should not be the only surviving outcome. Weak flaws elimination of Hillas and Samet (2020) deletes profiles $(B, R)$ (because if this profile is played, it means that strategy $R$ is played, implying that $B$ is weakly dominated), and then $(T, R)$ (after eliminating $(B, R)$, $R$ is strictly dominated by $L$). Outcomes $(T, L)$ and $(B, L)$ are surviving. Thus, there is no mutability. IERDS deletes no strategy. Now, we slightly modify the payoffs matrix in a way that yields order dependence of IEWDS outcome:

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | (2,1) | (3,1) |
| $B$ | (2,0) | (2,1) |

$\xrightarrow{\text{IEWDS}}$

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strat. | $L$ | $R$ |
| $T$ | (2,1) | (3,1) |

or

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strat. | $L$ | $R$ |
| $T$ | (2,1) | (3,1) |

Figure 4: Modified Game with IEWDS Order Dependence (and Mutability)

In this modified version of the game, a second pure Nash equilibrium appears: $(T, R)$. $T$ weakly dominates $B$ and $R$ weakly dominates $L$. If $B$ is eliminated, then $L$ is not dominated and both Nash equilibria survive. On the opposite, if $L$ is eliminated, so is $B$, and the only surviving outcome is $(T, R)$ (note that only the outcome $(T, R)$ can be achieved as well by eliminating $B$ and $L$ at the same step). There, IEWDS is order dependent. It may also generate mutability. Indeed, if the final outcome is $(T, R)$, then the strategy $L$ is not virtually dominated. Hillas and Samet (2020)'s deletion procedure eliminates both profiles $(B, R)$ and $(B, L)$, letting the two Nash equilibria survive. IERDS deletes no strategy.

Now, we focus on the last example of this section:

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | (2,0) | (3,1) |
| $B$ | (2,1) | (2,0) |

$\xrightarrow{\text{IEWDS}}$

| | $j$'s Strat. |
|---|---|
| $i$'s Strat. | $R$ |
| $T$ | (3,1) |

Figure 5: Modified Game with a Unique Prediction for IEWDS (and IERDS)

In this last version of the game, there are two Nash equilibria: $(B, L)$ and $(T, R)$. $T$ weakly dominates $B$. If $B$ is eliminated, then, $R$ dominates $L$ and the only outcome is $(T, R)$. There, IEWDS is not mutable. Indeed, since $R$ is played and thus uneliminated, $T$ does weakly dominate $B$. It is order independent as well. Moreover, it predicts a unique outcome whereas the Hillas and Samet (2020)'s procedure eliminates only the profile $(B, R)$, letting the two Nash equilibria survive. To compare IERDS to another procedure, notice that RBEU of Cubitt and Sugden (2011) accumulates the strategy $T$, but then stops[17]. IERDS deletes $B$, leading to the unique outcome $(T, R)$.

[17]Even if $T$ is played with a strictly positive probability, for all $j'$s beliefs where $B$ is played with a higher probability, $L$ is optimal and cannot be deleted.

To sum up, in these various examples, when weak dominance is mutable or order dependent, our elimination procedure deletes less strategy than IEWDS. When IEWDS is both non mutable and order independent, our elimination procedure predicts the same outcome as IEWDS (see Appendix E for an attempt to generalize this discussion in two-player games), being more predictive than weak flaws elimination of Hillas and Samet (2020) or RBEU of Cubitt and Sugden (2011).

Now, we state the result of this section, i.e., the immutability of IERDS. With the help of *sequences of games*, we recall what is immutability: a procedure is immutable if there is no process associated with it such that at the end of the sequence of game, there is no strategy $s'_i \in S^0_i \setminus S^\Lambda$ which is not dominated in the game formed by $\Gamma^\Lambda$ and the strategy $s'_i$, i.e. the game $\Gamma'$ (with the same players and utilities as $\Gamma^\Lambda$) and the strategy set $S^\Lambda \cup s'_i$. Now, we can state that there is no mutability in any sequence of games generated by IERDS:

**Theorem 2.** *IERDS is immutable in finite games.*

Proof is relegated to Appendix A.

## 7 Mixed root dominance

Consider the mixed extension of a game $\Gamma$ and denote $\Sigma$ the set $\prod_{i\in I}\Sigma_i \equiv \prod_{i\in I}\Delta(S_i)$ the set of all (mixed) strategies. Thus, $\sigma_i \in \Sigma_i$ is a mixed strategy if it is a probability distribution over the set $S_i$ of pure strategies. As in the pure strategy case, we denote $\Sigma_{-i}$ the set $\prod_{j\in I\setminus\{i\}}\Sigma_j \equiv \prod_{j\in I\setminus\{i\}}\Delta(S_j)$, the strategy profiles set of $i$'s opponents. Let $\sigma_i(s_i)$ be the probability that $s_i$ is effectively used when $\sigma_i$ is played and denote $R_{\sigma_i} = \{s_i \in S_i | \sigma_i(s_i) > 0\}$ the support of $\sigma_i$[18]. We apply the definition of a Best Reply Set to mixed strategies in the same way as in the pure strategy case:

**Definition 9.** *The Best Reply Set to $\sigma''_i \in \Sigma_i$, denoted $\boldsymbol{b}(\sigma''_i)$, is the set of all strategy profiles $\sigma^* \in \Sigma$ such that:*
*$\sigma^*_i = \sigma''_i$, and, if $S_{-i} \neq \emptyset$:*

$$\exists j \in I \setminus \{i\},\ \sigma^*_j \in \underset{\sigma_j\in\Sigma_j}{\arg\max}\ U_j(\sigma_j, \sigma^*_{-j}) \tag{OM$'$}$$

Now, we extend the notion of Best Reply Set to strategy subsets:

[18]Note that this definition of the support cannot be weakened by allowing e.g. a continuous distribution as a support. We clarify this point below.

**Definition 10.** *For any strategy subset $\bar{S}_i \subset S_i$, we denote $\boldsymbol{b}(\bar{S}_i) = \bigcup_{\sigma_i \in \Delta \bar{S}_i} \boldsymbol{b}(\sigma_i)$ the Best Reply Set to the strategy subset $\bar{S}_i$.*

Note that if the subset is a singleton, Definitions 3 and 10 obviously coincide. Importantly, in order to define *mixed root dominance*, we will use the Best Reply Set to the strategy subset formed by the support of the mixed strategy:

**Definition 11.** *A strategy $s'_i \in S_i$ is said root dominated by the mixed strategy $\sigma''_i \in \Sigma_i$ whose support is $R_{\sigma''_i}$, if:*

$$\forall s_{-i} \in S_{-i}: \qquad U_i(\sigma''_i, s_{-i}) \geq U_i(s'_i, s_{-i}) \tag{RD1$'$}$$

$$\forall \sigma^*_{-i} \text{ such that } \sigma^* \in \boldsymbol{b}(R_{\sigma''_i}): U_i(\sigma''_i, \sigma^*_{-i}) > U_i(s'_i, \sigma^*_{-i}) \tag{RD2$'$}$$

Definition 11 is in fact a generalization of Definition 4. Besides, if the Best Reply Set to $\sigma''_i \in \Sigma_i$ was defined such that it contained only the best responses to $\sigma''_i$, root dominance would lack hereditariness. Assume a mixed strategy $\sigma''_i \in \Sigma_i$ composed of two pure strategies in $S_i$, $s''_i$ and $s'_i$ such that $\sigma''_i \underset{\text{S}}{\succ} s'_i$. Then, it is immediate that $s''_i$ strictly dominates $s'_i$. Here, the point is that the mixed strategy is not necessary to establish that $s'_i$ is strictly dominated: that is, even if $\sigma''_i$ is eliminated, $s''_i$ still strictly dominates $s'_i$[19]. Concerning root dominance, the fact is that mixing does not affect only the payoffs, it affects also the set of best responses. In order to keep hereditariness, all the best responses to strategies contained in $\Delta(R_{\sigma''_i})$ have to be considered. We can see it with the next example, where we use directly Definition 4 to define root dominance by mixed strategies and not Definition 11:

| | $j$'s Strategy | | |
|---|---|---|---|
| $i$'s Strat. | $L$ | $C$ | $R$ |
| $T$ | $(\underline{4},\underline{0})$ | $(\underline{4},\underline{0})$ | $(0,\underline{0})$ |
| $M$ | $(\underline{4},\underline{0})$ | $(\underline{4},\underline{0})$ | $(4,\underline{0})$ |
| $B$ | $(0,0)$ | $(\underline{4},\underline{2})$ | $(\underline{8},1)$ |

with dominance relations:

| | |
|---|---|
| $R$ | $\succ L$ |
| $\sigma_R$ | $\succ L, R$ |
| $C$ | $\not\succ L, R$ |

Figure 6: Order Dependence Issue with Definition 4 applied to Mixed Strategies

Strategy $C$ weakly dominates both $L$ and $R$. However, it does not root dominate them, player $i$ best responding to $C$ with the three strategies $T$, $M$ and $B$. Now, look

[19] A different property but implying similar consequences is established for root dominance in Lemma 7.

at any mixing $\sigma_R$ of $C$ and $R$. Then, $i$'s best response is only $B$. Instead, for any mixing $\sigma_L$ of $C$ and $L$, $i$'s best responses are $T$ and $M$.

Thus, if $\boldsymbol{b}(\sigma_R) = (\sigma_R, B)$, $\sigma_R$ root dominates $R$. Besides, $R$ root dominates $L$. On the opposite, if $\boldsymbol{b}(\sigma_L) = \{(\sigma_L, T), (\sigma_L, M)\}$, $\sigma_L$ does not root dominates $L$. Therefore, both $L$ and $R$ are root dominated but only $R$ root dominates $L$. Consequently, if $R$ is eliminated before $L$, $L$ cannot be eliminated at any further step, showing that the procedure would be order dependent. When we apply Definition 11, the dominance relation is modified such that:

$$\sigma_R \not\succ R \text{ and } \sigma_R \not\succ L$$

Then, order dependence disappears and only strategy $L$ is eliminated.

The next result states that mixed strict dominance implies mixed root dominance:

**Lemma 6.** $\sigma_i'' \underset{\mathrm{S}}{\succ} s_i \Rightarrow \sigma_i'' \succ s_i'$.

The proof is straightforward since strict dominance implies trivially both RD1′ and RD2′.

The next example shows how mixed IERDS behaves with respect to pure IERDS. It presents the final outcome associated to each procedure. Assume a Bertrand duopoly where the marginal cost is zero, the market size equal to 1 and admit as classically that when both firms set the same price, the market is equally shared. Then we have following payoffs matrix:

| | $j$'s Strategy | | | | |
|---|---|---|---|---|---|
| $i$'s S. | 0 | 1 | 2 | 3 | 4 |
| 0 | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) |
| 1 | (0,0) | (0.5,0.5) | (1,0) | (1,0) | (1,0) |
| 2 | (0,0) | (0, 1) | (1, 1) | (2,0) | (2,0) |
| 3 | (0,0) | (0,1) | (0, 2) | (1.5,1.5) | (3,0) |
| 4 | (0,0) | (0,1) | (0,2) | (0,3) | (2,2) |

$\xrightarrow{}$ pure IERDS

| | $j$'s Strategy | | | |
|---|---|---|---|---|
| $i$'s S. | 1 | 2 | 3 | 4 |
| 1 | (0.5,0.5) | (1,0) | (1,0) | (1,0) |
| 2 | (0, 1) | (1, 1) | (2,0) | (2,0) |
| 3 | (0,1) | (0, 2) | (1.5,1.5) | (3,0) |
| 4 | (0,1) | (0,2) | (0,3) | (2,2) |

Figure 7: Symmetric Discrete Bertrand Game after Pure IERDS

Once the strategies 0 are eliminated, no strategy is any longer root dominated by a pure one. However, one can find a mixture of strategies 1, and 3 that root dominates the strategy 4 (it is enough to have a weight higher than $\frac{2}{3}$ for strategy 3 and a strictly positive weight for 1). After elimination of the strategies 4, some mixtures of strategies 1 and 2 can strictly dominates the strategy 3 (it is enough to have a weight higher than $\frac{1}{2}$ for strategy 2 and a strictly positive weight for 1). Once strategies 3 are eliminated, we can show that strategies 2 are root dominated by 1.

| | $j$'s Strategy | | | | |
|---|---|---|---|---|---|
| $i$'s S. | 0 | 1 | 2 | 3 | 4 |
| 0 | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) |
| 1 | (0,0) | (0.5,0.5) | (1,0) | (1,0) | (1,0) |
| 2 | (0,0) | (0, 1) | (1, 1) | (2,0) | (2,0) |
| 3 | (0,0) | (0,1) | (0, 2) | (1.5,1.5) | (3,0) |
| 4 | (0,0) | (0,1) | (0,2) | (0,3) | (2,2) |

$\xrightarrow[\text{mixed IERDS}]{}$

| | $j$'s S. |
|---|---|
| $i$'s S. | 1 |
| 1 | (0.5,0.5) |

Figure 8: Symmetric Discrete Bertrand Game after Mixed IERDS

The next result demonstrates that a strategy $s'_i \in S_i$ root dominated by a mixed strategy whose support contains $s'_i$ is also root dominated by another strategy whose support does not contain $s'_i$:

**Lemma 7.** *If $s'_i \in S_i$ is root dominated by $\sigma''_i \in \Sigma_i$ such that $R_{\sigma''_i} = (\check{S}_i \cup s'_i)$ (with $\check{S}_i \subset S_i$ which contains at least one strategy different from $s'_i$), then $s'_i$ and $\sigma''_i$ are root dominated by $\sigma'''_i \in \Sigma_i$ such that $R_{\sigma'''_i} = \check{S}_i$.*

*Proof.* Assume $s'_i \in S_i$ is root dominated by $\sigma''_i \in \Sigma_i$. Thus, $\sigma''_i$ weakly dominates $s'_i$. Then, we can construct $\sigma'''_i \in \Sigma_i$ such that the weight of each pure strategy forming $\sigma'''_i$ is proportionally the same as in $\sigma''_i$ when $s'_i$ is removed from the support. It is clear that $\sigma'''_i$ weakly dominates $\sigma''_i$ (and $s'_i$). Indeed, the average payoff is (weakly) increased when $s'_i$ is removed, since the payoff to $i$ of $s'_i$ against any profile is below the average payoff of $\sigma''_i$. Furthermore, we know that $\sigma''_i$ strictly payoff dominates $s'_i$ on $\boldsymbol{b}(R_{\sigma''_i})$. For the same reason, it is clear that $\sigma'''_i$ strictly payoff dominates $\sigma''_i$ (and $s'_i$) on $\boldsymbol{b}(R_{\sigma''_i})$. By construction, $\boldsymbol{b}(R_{\sigma'''_i}) \subset \boldsymbol{b}(R_{\sigma''_i})$. Therefore, $\sigma''_i$ (and $s'_i$) are root dominated by $\sigma'''_i$. ■

This result allows us to keep the result of Lemma 2, and then to show order independence of mixed IERDS:

**Theorem 3.** *Mixed IERDS is order independent.*

*Proof.* By adding Lemma 7, all results of Section 4 hold true when we apply the mixed framework. See Appendix G for more details. ■

Finally, we can establish the next statement:

**Theorem 4.** *Mixed IERDS refines mixed IESDS.*

This result is a direct implication of Lemma 6 and Theorem 3.

To be clear this result means that in some games, mixed IERDS refines strictly mixed IESDS and in other games, they have the same outcome.

In the next section, we will simply write IERDS for mixed IERDS.

# 8 Rationality concepts

First, we recall that in the standard framework, in a two-player game, a strategy $s_i^*$ is rational if and only if there is a strategy for $j$ such that $\sigma_i^*$ maximizes the utility of $i$:

**Definition 12.** *A strategy $s_i^*$ is rational if:*

$$\begin{aligned}&\exists \sigma_j \in \Sigma_j, \ \textit{such that } \forall \sigma_i \in \Sigma_i,\\&U_i(s_i^*, \sigma_j) \geq U_i(\sigma_i, \sigma_j)\end{aligned}$$

By Pearce (1984, Lemma 3), in a two-player game, a strategy is not rational if and only if it is strictly dominated. In the remaining of the paper, we will restrict ourselves to two-player games as well. Furthermore, Pearce (1984, Lemma 4) shows that a strategy is weakly dominated if and only if it is not a best response to any totally mixed profile. That is, beliefs are said *cautious*,i.e., players believes that opponents plays only full support strategies). This cautiousness is justified by the fact that players may not exclude totally the possibility that opponents can play any strategy. Yet, this view is apparently in contradiction with the belief that weakly dominated strategies should not be played. Indeed, admissibility requires that agents consider possible that opponents play all their strategies with positive probability. It means that each agent believes that his opponents will play non admissible strategies. This is emphasized by Samuelson (1992) as the third issue with IEWDS:

> The process appears initially to call for agents to assume that opponents may play any of their strategies but subsequently to assume that opponents will certainly not play some strategies.

This problem is known as the *inclusion-exclusion challenge* (see Barelli and Galanis (2013)) and has opened a rich literature attempting to reconcile weak dominance rationality with consistency.

Brandenburger (1992); Stahl (1995); Brandenburger et al. (2008) propose to use the *lexicographic probability system* introduced in Blume et al. (1991) to characterize weak dominance rationality. In a word, it is assumed players believe that when a strategy is eliminated, it is infinitely less likely to be played with respect to remaining strategies, but still infinitely more likely to be played than previously eliminated strategies. Therefore, the inclusion-exclusion challenge is solved in an elegant way: a weakly dominated strategy is unlikely to be played and at the same time not totally unlikely if "necessary". In contrast with the view defended in Samuelson (1992), Brandenburger et al. (2008) state that: "A player is rational if he optimizes and also rules nothing out." Alternatively, Barelli and Galanis (2013) introduce the notion of *event-rationality* which allows two levels of beliefs. A first which is standard, and a second one used in case of equivalence between two strategies. When there is equivalence, a player can break ties by using opponents' strategies deemed unlikely. Therefore, again, even dominated strategies are never totally excluded of the players' "thoughts". The rationality concepts we introduce do consider thought experiments but contrasts with the option proposed in Barelli and Galanis (2013): our experiments assume a certain sense of rationality about the opponent's play at the second level of belief.

In the next subsection, we will assume that some perturbations of the game can occur with probability $\epsilon > 0$. Considering ruling out "unreasonable" Nash equilibria in extensive-form games, Selten (1975) formalizes this idea with the notion of *perfect equilibria*, which are Nash equilibria robust to the possibility that agents may deviate (by mistakes). Additionally, Fudenberg et al. (1988) introduce the idea that payoffs knowledge might not be complete, i.e., agents are unsure about their own payoffs and others' payoffs. Therefore, the authors introduce forward induction in the reasoning: the deviation is not necessarily a mistake but might be a "signal". The DF procedure of Dekel and Fudenberg (1990) is the outcome of such games where agents are uncertain about payoffs. Besides, Börgers (1994) shows that the DF procedure can also be the result of *approximate* common knowledge of weak dominance rationality (i.e. each player believes that his opponents play strategies with full support). That is, Börgers (1994) assumes that weak dominance rationality is common knowledge with probability $p$. When $p$ converges to 1, agents plays only strategies which remain after the DF procedure.

The kind of perturbations we introduce does not consider such payoff uncertainties. Rather, we are closer to Selten (1975)'s idea that a player may observe "mistakes" and react optimally. We also relate to Hamilton and Slutsky (2005) who study the possibility that an agent takes into account his own errors. More precisely, we consider simultaneous games where an agent can generate reactions by his own thoughts. We suppose that despite having a "reference" strategy, a player may alternatively consider some strategy subsets with probability $\epsilon$. If so, the opponent reacts optimally (in a naive way) to this strategy subset. Thus, the reference strategy is "tested" against such mind trembles. "Mind trembles" can be seen as potential trembles which will be realized only if they are profitable. For instance, assume a poker player who sets a reference strategy before the game starts. However, he knows that during the play he may be tempted to adopt another strategy with probability $\epsilon$: this is a mind tremble. Now, if the reference strategy is not optimal when he believes that the opponent can detect this tremble and react optimally, the tremble should be realized and, in fact the reference strategy never played in such a game.

Therefore, we assume a framework with *conjectural variations* (see our discussion below in Section 8.3). Then, we introduce the concept of *local $\epsilon$-rationality* which selects the strategies maximizing $i$'s utility when $i$ forms conjectures about $j$'s reactions to mind trembles, those occurring with probability $\epsilon$. With respect to the usual conjectural variation framework, two differences operate: $(i)$ an actual deviation is not required but a mind tremble is enough to generate the opponent's reaction, and $(ii)$ reaction is said *rational*, i.e., agent $i$ conjectures that $j$ will play a best response to the mind tremble.

In Appendix I we propose two others perturbations that lead to two additional characterization of root undominance by rationality.

## 8.1 Characterization of root undominance by rationality

In order to characterize root undominance, we define in this subsection a new rationality concept called *local $\epsilon$-rationality*. For this purpose, we introduce first a *conjectural system* $C_{ij}$ for player $i$ about strategies of player $j$ when a perturbation occurs (with probability $\epsilon$). We say that player $i$ has a *mind tremble* when he thinks of a strategy subset $\check{S}_i \subset S_i$ whereas he has a *reference* strategy $\sigma_i \in \Sigma_i$. Finally, for each strategy subset $\check{S}_i \subset S_i$, $i$ believes that $j$ will play a certain strategy $s_j$ with probability $C_{ij}\left(\check{S}_i, s_j\right)$ if $i$ has a mind tremble towards $\check{S}_i$.

We define $C_{ij}$ as a mapping from the tuple formed by the product of the power set $\mathcal{P}(S_i)$[20] of $S_i$ and $j$'s strategy set $S_j$ to $[0,1]$. Our conjectural system is naturally reminiscent of the conjectural variation theory (see our discussion below in Section 8.3), except that we consider strategy subsets.

**Definition 13.** *A conjectural system $C_{ij}$ is the mapping $C_{ij} := \mathcal{P}(S_i) \times S_j \to [0,1]$ which associates any $i$'s strategy subset with a pure strategy for $j$ to a probability, that*

[20]The power set of $S_i$ is the set containing all the subsets of $S_i$.

*is, it satisfies:* $\forall \check{S}_i \in \mathcal{P}(S_i),\ \sum_{s_j \in S_j} C_{ij}\left(\check{S}_i, s_j\right) = 1.$

According to $i$'s belief, $C_{ij}\left(\check{S}_i, s_j\right)$ is the probability that $j$ will play $s_j$ if he "observes" that $i$ thinks of a strategy whose support is $\check{S}_i$. We denote $\mathcal{C}_{ij}$ the set of all conjectural systems of $i$ about $j$.

Before going further, we define an additional notion which will be useful below, namely, the *expected $\epsilon$-perturbed utility*:

**Definition 14.** *The expected $\epsilon$-perturbed utility of $i$ from playing $\sigma_i$ when $j$ plays $\sigma_j$ in the game without perturbation and $\widehat{\sigma}_j$ in the game with perturbation is:*

$$V_i^\epsilon(\sigma_i, \sigma_j, \widehat{\sigma}_j) \equiv (1-\epsilon)\,\mathbb{E}[U_i(\sigma_i, \sigma_j)] + \epsilon\,\mathbb{E}[U_i(\sigma_i, \widehat{\sigma}_j)]$$

Simply, the $\epsilon$-perturbed utility formalizes the expected utility when player $i$ has the belief that $j$ plays $\sigma_j$ with probability $1-\epsilon$ and $\widehat{\sigma}_j$ with probability $\epsilon$. In the remainder of the paper, we will see $\sigma_j$ as the "normal" or standard belief (the belief when no exogenous event occurs), and we will assume that $\widehat{\sigma}_j$ is played when a perturbation occurs. We emphasize that it does not mean that an extensive form game is played. Rather, the thoughts of $i$ (about his own play) influence his beliefs about $j$'s actions with probability $\epsilon$.

For a given mixed strategy $\sigma_i \in \Sigma_i$ of player $i$, we recall that we denote $\sigma_i(s_i)$ the probability that $s_i$ to be drawn when $\sigma_i$ is chosen. Now, we introduce a new rationality concept in association to a conjectural system:

**Definition 15.** *A strategy $s_i \in S_i$ is locally $\epsilon$-rational if:*

$$\begin{aligned}
&\exists \sigma_j \in \Sigma_j, \exists C_{ij} \in \mathcal{C}_{ij},\ \textit{such that}\ \forall \sigma_i \in \Sigma_i,\ \textit{if we set:}\\
&\sigma_j^*\ \textit{with}\ \sigma_j^*(s_j) \equiv C_{ij}\left(R_{\sigma_i}, s_j\right) \textit{then we have,}\\
&V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)
\end{aligned}$$

From now, distinctly from the conjectural variation theory, we assume that the conjectures are *rational* (see our discussion below in Section 8.3), i.e., $C_{ij}\left(\check{S}_i, s_j\right)$ cannot be strictly positive unless $s_j \in \boldsymbol{b}(\check{S}_i)$:

**Assumption R.** *$C_{ij}$ is a rational conjectural system (with $\mathcal{R}_{ij}$ the set of such rational conjectural systems), i.e.:*

$$\forall \left(\check{S}_i, s_j\right) \in \mathcal{P}(S_i) \times S_j, C_{ij}\left(\check{S}_i, s_j\right) > 0 \Rightarrow s_j \in \boldsymbol{b}(\check{S}_i)$$

Now, we can re-write our definition:

**Definition 16.** *Under* Assumption **R**, *a strategy $s_i \in S_i$ is locally $\epsilon$-rational if:*

$$\begin{aligned}
&\exists \sigma_j \in \Sigma_j, \exists C_{ij} \in \mathcal{R}_{ij}, \text{ such that } \forall \sigma_i \in \Sigma_i, \text{ if we set:}\\
&\sigma_j^* \text{ with } \sigma_j^*(s_j) \equiv C_{ij}\left(R_{\sigma_i}, s_j\right) \text{ then we have:}\\
&V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)
\end{aligned}$$

Under Assumption **R**, a strategy $s_i \in S_i$ is locally $\epsilon$-rational if there is a belief $\sigma_j$ and a rational conjectural system $C_{ij}$ such that the expected utility of $s_i$ is larger than any tested $\sigma_i \in \Sigma_i$. That is, $s_i$ is optimal if $i$ believes that $j$ plays $\sigma_j$ with probability $1-\epsilon$ and reacts optimally to the tested $\sigma_i$ with probability $\epsilon$.

Naturally, we can establish the following result that simplifies the previous definition:

**Lemma 8.** *Under* Assumption **R**, *a strategy $s_i \in S_i$ is locally $\epsilon$-rational if and only if it verifies:*

$$\exists \sigma_j \in \Sigma_j, \text{ such that } \forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i}), \quad V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \qquad (1)$$

*Proof.* Assume $s_i$ is locally $\epsilon$-rational. Then, there is $\sigma_j \in \Sigma_j$ and a rational conjectural system $C_{ij}$ against which $s_i$ (weakly) payoff dominates all other strategies in $\Sigma_i$. That is, if we compare $s_i$ to any $\sigma_i \in \Sigma_i$, we use with probability $1-\epsilon$ the strategy $\sigma_j$ and with probability $\epsilon$ the strategy $\sigma_j^*$ such that $\sigma_j^*(s_j) \equiv C_{ij}\left(R_{\sigma_i}, s_j\right)$. By Assumption **R**, we know that all $s_j$ are in $\boldsymbol{b}(R_{\sigma_i})$. Therefore, $\sigma_j^*$ is in $\boldsymbol{b}(R_{\sigma_i})$. Finally, we can write that:

$$\begin{aligned}
&\exists \tilde{\sigma}_j \in \Sigma_j, \forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i}) \text{ such that:}\\
&V_i^\epsilon(s_i, \tilde{\sigma}_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \tilde{\sigma}_j, \sigma_j^*)
\end{aligned} \qquad (1)$$

Conversely, assume the above Equation (1). If this is true we can construct a rational conjectural system $C_{ij}$ by using the hyperplane theorem. Assume a strategy subset $\check{S}_i \in \mathcal{P}(S_i)$. Consider the vectors $\overrightarrow{V_i^\epsilon}(\sigma_i, \check{S}_i) = \{V_i^\epsilon(\sigma_i, s_j, s_j^*)\}_{s_j \in S_j, s_j^* \subset \boldsymbol{b}(\check{S}_i)}$[21] for each $\sigma_i \in s_i \cup \Delta(\check{S}_i)$. Simply, these vectors are such that each component $l+m$ is the payoff $i$ can obtain when playing $\sigma_i$ and when $j$ plays the pure strategy $s_j^l \in S_j$ with probability $1-\epsilon$ and the pure strategy $s_j^{*^m} \subset \boldsymbol{b}(\check{S}_i)$ with probability $\epsilon$. We denote $Y(s_i, \check{S}_i)$ the set of such vectors. Besides, we can construct the following set $X$. If $k$ is equal to $\sharp(S_j) \times \sharp(\boldsymbol{b}(\check{S}_i))$[22], then $X$ is the set $\left\{x \in \mathbb{R}^k \,|x > \overrightarrow{V_i^\epsilon}(s_i)\right\}$, that is the set of

[21] Note that we make a slight abuse of notation here: we consider $s_j^* \subset \boldsymbol{b}(R_{\sigma_i})$ if $\exists \sigma_j \subset \boldsymbol{b}(R_{\sigma_i})$ and $s_j \in R_{\sigma_i}$. For technical reasons, we consider only pure strategies but all mixed strategies in the Best Reply Set are well present through the pure strategies that support them.

[22] We denote $\sharp(S_i)$ the number of elements in the set $S_i$. According to the above footnote, $\sharp(\boldsymbol{b}(\check{S}_i))$ is well finite.

all payoffs that strictly dominate $s_i$ payoffs. Both $X$ and $Y(s_i, \check{S}_i)$ are convex sets. By Equation (1), these sets are disjoint. Then, we can apply the separating hyperplane theorem which states that there is a vector in $\mathbb{R}^k$, $\pi \geq 0$ with $\pi \neq 0$ and such that:

$$\forall y \in Y(s_i, \check{S}_i), \forall x \in X, \pi \cdot y \leq \pi \cdot \overrightarrow{V_i^\epsilon}(s_i) \leq \pi \cdot x$$

It directly implies that $\forall \sigma_i \in s_i \cup \Delta(\check{S}_i),\ \pi \cdot \left(\overrightarrow{V_i^\epsilon}(s_i) - \overrightarrow{V_i^\epsilon}(\sigma_i)\right) \geq 0$.

Especially, there is such a vector $\tilde{\pi}$ such that $\forall l \in [[1, \sharp(S_j)]],\ \tilde{\pi}(l) = (1-\epsilon) \times \tilde{\sigma_j}(\sigma_j^l)$, since the hypothesis that Equation (1) is verified implies (by continuity of $V_i^\epsilon$ in $\epsilon$) that:

$$\forall \sigma_i \in \Sigma_i,\ U_i(s_i, \tilde{\sigma_j}) \geq U_i(\sigma_i, \tilde{\sigma_j})$$

Thus we can have $\forall \sigma_i \in s_i \cup \Delta(\check{S}_i),\ \tilde{\pi} \cdot \left(\overrightarrow{V_i^\epsilon}(s_i) - \overrightarrow{V_i^\epsilon}(\sigma_i)\right) \geq 0$ when $\epsilon \to 0^+$.

Now, we can start constructing the rational conjectural system $C_{ij}$ by setting

$$\forall s_j^{*^m} \subset \boldsymbol{b}(\check{S}_i),\ C_{ij}(\check{S}_i, s_j^{*^m}) \equiv \pi(\sharp(S_j) + m)$$

It is clear that it is rational since $s_j^{*^m} \subset \boldsymbol{b}(\check{S}_i)$ We can apply all the previous reasoning to each $\check{S}_i \in \mathcal{P}(S_i)$ with $\forall \check{S}_i \in \mathcal{P}(S_i), \forall l \in [[1, \sharp(S_j)]],\ \tilde{\pi}(l) = (1-\epsilon) \times \tilde{\sigma_j}(\sigma_j^l)$. Finally, we obtain a full rational conjectural system and we can write that:

$$\begin{aligned}
&\exists \tilde{\sigma_j} \in \Sigma_j, \exists C_{ij} \in \mathcal{R}_{ij},\ \text{such that } \forall \sigma_i \in \Sigma_i,\ \text{if we set:}\\
&\sigma_j^* \text{ with } \sigma_j^*(s_j) \equiv C_{ij}\left(R_{\sigma_i}, s_j\right) \text{then we have:}\\
&V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)
\end{aligned}$$

■

Now, we can state the main result of this section, the characterization of root undominance by local $\epsilon$-rationality:

**Theorem 5.** *Under Assumption* **R**, *a strategy $s_i \in S_i$ is locally $\epsilon$-rational when $\epsilon \to 0^+$ if and only if it is root undominated.*

*Proof.* Assume Equation (1) for $s_i$ and by contrapositive that $s_i$ is root dominated. Therefore $\exists \sigma_i'' \in \Sigma_i$ such that $\forall \sigma_j \in \Sigma_j, U_i(\sigma_i'', \sigma_j) \geq U_i(s_i, \sigma_j)$ (RD1′), and $\forall \sigma_j^* \in \boldsymbol{b}(R_{\sigma_i''}), U_i(\sigma_i'', \sigma_j^*) > U_i(s_i, \sigma_j^*)$ (RD2′). Then, clearly:

$$\begin{aligned}
&\forall \epsilon > 0,\ \forall \sigma_j \in \Sigma_j, \forall \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i''})\\
&V_i^\epsilon(\sigma_i'', \tilde{\sigma_j}, \sigma_j^*) > V_i^\epsilon(s_i, \tilde{\sigma_j}, \sigma_j^*)
\end{aligned}$$

It is an immediate contradiction with Equation (1). By Lemma 8, $s_i$ is not locally $\epsilon$-rational.

Now assume that $s_i$ is root undominated. Then, there is no $\sigma_i \in \Sigma_i$ such that both $\forall \sigma_j \in \Sigma_j,\ U_i(\sigma_i, \sigma_j) \geq U_i(s_i, \sigma_j)$ (RD1′) and $\forall \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i}),\ U_i(\sigma_i, \sigma_j^*) > U_i(s_i, \sigma_j^*)$ (RD2′). Then, for all $\sigma_i$ where Equation (RD1′) is not respected, $\exists \sigma_j$ such that $\forall \sigma_j^* \in \boldsymbol{b}(R_{\sigma_i}),\ V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)$ when $\epsilon \to 0^+$. As well, for all $\sigma_i$ where Equation (RD2′) is not respected, $\exists \sigma_j^* \in \boldsymbol{b}(R_{\sigma_i})$ such that $V_i^\epsilon(s_i, \sigma_j^*, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j^*, \sigma_j^*)$. Thus, we can say that $\forall \sigma_i \in \Sigma_i, \exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \in \boldsymbol{b}(R_{\sigma_i})$ such that:

$$V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)$$

Again, we can use the separating hyperplane theorem to show there is in fact a $\sigma_j \in \Sigma_j$ such that $\forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \in \boldsymbol{b}(R_{\sigma_i})$ such that:

$$V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \tag{1}$$

By Lemma 8, $s_i$ is locally $\epsilon$-rational.

■

Remark that the expression $\epsilon \to 0^+$ implies that for a given game, $\exists \bar{\epsilon} > 0$ such that $\forall \epsilon < \bar{\epsilon}$, there is equivalence between root undominance and local $\epsilon$-rationality.

## 8.2 Other notions of rationality

In this section we introduce different but close notions of rationality with respect to the one introduced in the previous subsection. It will help us to understand what local $\epsilon$-rationality is and is not. As well, it will be useful in the following subsections. We distinguish local $\epsilon$-rationality from global $\epsilon$-rationality and self-local $\epsilon$-rationality. Global $\epsilon$-rationality induces the belief that the strategy support of the strategy actually played is observed by the opponent with probability $\epsilon$. Instead, self-local $\epsilon$-rationality is such that $i$ believes that $j$ observes the strategy support of the reference strategy. We can summarize these differences in Table 2. Two main differences appear: first, what the agent conjectures his opponent may observe if he detects the agent's thoughts. Second, which kind of utility is maximized for each rationality concept. Self-local $\epsilon$-rationality is an ex post concept because once the opponent believes the agent is committed to a given strategy, the agent can still decide to move ex post. By contrast, the global concept is ex ante since the strategy since once the agent is committed to an action, he cannot move. Finally, local $\epsilon$-rationality correspond to a *projected* utility

| Rationality | Type of Utility Maximization | Observed Strategy |
|---|---|---|
| Global | Ex-Ante (Conjectural Variation) | Played Strategy |
| Self-local | Ex-Post | Reference Strategy |
| Local | Projected | Targeted Strategy |

Table 2: Rationalities

maximization. That is, even if the agent might play his reference strategy, testing it against other strategies induces beliefs such that this reference strategy was not available anymore. Alternatively, we can remark that our three notions of rationality can be interpreted and distinguished with the conjectures about the opponent's speed of adjustment. Self-local rationality corresponds to the case where the agent conjectures that his opponent his stickier and is not able to adjust his strategies. Then, opponents best respond to the reference (considered initially) strategy. Global rationality corresponds to the case where the opponent adjusts perfectly and then always best responds. Local rationality is such that the agent conjectures that his opponent anticipates the adjustment, even if no move is finally made. That is why the opponent only best responds to the targeted strategy.

Now, we define our two additional concepts:

**Definition 17.** *A strategy $s_i \in S_i$ is globally $\epsilon$-rational if and only if:*

$$\begin{aligned}&\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \boldsymbol{b}(s_i), \forall \sigma_i \in \Sigma_i, \exists \sigma_j^{**} \subset \boldsymbol{b}(R_{\sigma_i}) \text{ such that:}\\ &V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^{**})\end{aligned}$$

Global rationality of $s_i$ means that $s_i$ may maximize the ex-ante utility of $i$, given that whatever the strategy chosen by $i$, $j$ reacts optimally to it with probability $\epsilon$.

**Definition 18.** *A strategy $s_i \in S_i$ is self-local $\epsilon$-rational if and only if:*

$$\begin{aligned}&\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \boldsymbol{b}(s_i), \text{ such that } \forall \sigma_i \in \Sigma_i,\\ &V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)\end{aligned}$$

Self-local rationality of $s_i$ is the converse of local $\epsilon$-rationality of $s_i$ in terms of reference point. That is, when $i$ considers the strategy $s_i$, he believes that $j$ reacts optimally to $s_i$ with probability $\epsilon$. The strategy is self-local rational if there is a belief satisfying this condition such that no move increases the $i$'s payoff. In other words, $s_i$ may maximize $i$'s ex post utility given that $s_i$ is the reference point to which $j$ best responds with probability $\epsilon$.

Besides, note that these three notions of rationality are obviously a refinement of the classical one (see Definition 12) when $\epsilon \to 0^+$:

**Fact 1.** *A strategy is globally/self-locally/locally $\epsilon$-rational when $\epsilon \to 0^+$ only if it is rational.*

## 8.3 Links with the conjectural variation theory

Since Bowley (1924, p.38) has introduced the idea of conjectural variations, the industrial organization literature has largely been fueled by this theory[23] which considers that a market situation can remain stable if the conjectures all firms have on their opponents refrain all of them from deviating. Contrary to the Cournot approach, the equilibrium does not emerge from a *tatônnement*, but is postulated ex ante. The interest is to understand why competitors may not deviate from a situation far from a Cournot equilibrium. For instance, Sweezy (1939) introduces the kinked demand by arguing that firms react differently when they face opponents' downward or upward price moves. Nevertheless, conjectures can be insane and consequently sustain an infinite number of conjectural variation equilibria. That is why several authors had tried to *rationalize* the agents' conjectures. Notably, they stated that the conjectured reactions should be "optimal" in a certain sense (see for example Hahn (1978); Laitner (1980); Ulph (1983)). Mainly, the conjecture of player $i$ should be such that he expects that his opponent $j$ maximizes his utility given $j$'s conjectures (i.e. $j$ anticipates the reaction of $i$ after his own deviation responding to $i$'s deviation), once he has attained the new "statu quo". Yet, these attempts have been showed to miss their mark. Strikingly, Makowski (1987) notices two main problems with the concepts developed in the papers cited above. The first one is that the reaction of the opponent is optimal with respect to the new "statu quo", and not from the initial equilibrium. In other words, an agent does not conjecture that an opponent who faces his deviation will best respond to the deviation, simply that once he has moved to the new equilibrium, he does not wish to move (but the move is not *rationalized*). Alternatively, Makowski (1987) proposes to consider this type of conjecture with best responses to the deviation with the notion of *only slightly more rational, rational conjecture equilibrium* or SMR-RCE. However, he points out himself another flaw: conjectures are not time consistent. That is, when player $i$ maximizes his utility, he considers his potential deviation followed by the reaction of his opponent $j$. And $j$ maximizes his utility by considering also that his potential deviation will be followed by the reaction of his opponent $i$. In words of Makowski (1987), $i$ expects that the game ends at time $t = 2$ (after $j$'s response to his

[23]See e.g. Figuières et al. (2004) for a review. Besides, for a recent contribution of this theory to public economics, see McGinty (2021).

deviation), but conjectures as well that $j$ expects that the game ends at time $t = 3$ (after $i$'s response to $j's$ response). In fact, we can simply observe that there is no reason that the process stops at any given time. Undeniably, with SMR-RCE, $i$ does not consider he can deviate from the new "statu quo" he will establish by deviating a first time (whereas he may naturally want to deviate if he has a better response to the new "statu quo"). This criticism might seem severe, since many concepts[24] assume an end in the reasoning process when a deviation is tested. However, this criticism generally vanishes when the agents react by playing best responses, ending *de facto* the reasoning process of the deviator once an equilibrium is reached (if it exists of course). If each reaction is conjecture dependent, the next reaction is conjecture dependent as well. If a reaction is not based on arbitrary conjectures, but solely on optimality, then the reasoning process may terminate immediately.

Clearly, the beliefs assumed under global $\epsilon$-rationality have the flavor of an "$\epsilon$-rational conjectural variation". The previous discussion shows the trouble with two players "behaving in the same way"[25]. That is, if the deviator $i$ believes that $j$ will react optimally, there could be a difficulty if $j$ believed that $i$ will best respond in turn. This problem is technically solved when $\epsilon$ converges to 0, since it becomes obvious that $i$ should not move (ex-ante) in reaction to the conjectured response of $j$ which can only occur with a small probability. The meaning of such a theory when $\epsilon$ moves away from zero is an open question. We attempt to give some answers in Appendix J.

Before this, how to situate local $\epsilon$-rationality in this framework? Local $\epsilon$-rationality seems to be the converse of a $\epsilon$-*rational* conjectural variation theory. $\epsilon$-*Rational* conjectural variation could be stated (partially) as follows: if player $i$ deviates, $j$ will react optimally with probability $\epsilon$. Now, local $\epsilon$-rationality states that: whether player $i$ deviates or not, $j$ will react optimally to the deviation with probability $\epsilon$. Thus, why would $j$ reacts to a deviation that may not appear? Why would it be more reasonable? We attempt to answer these questions in the next subsections.

## 8.4 Observability of actions

Hamilton and Slutsky (1990) consider a duopoly where firms can choose the timing of their action before playing the actual game. That is, a firm can decide to move at the first period. In this case, if the competitor does not move first as well, the game is a Stackelberg duopoly (i.e. the follower observes the action at the first pe-

[24]Simply, think of the ones introduced in this paper and other as the intuitive criterion of Cho and Kreps (1987) (see our discussion below in Section 8.5).

[25]Assuming asymmetry seems justifiable since the deviatior decides alone to deviate and then, introduce asymmetry *de facto*.

riod). Otherwise, the game is simultaneous (e.g., it becomes a Cournot game if the considered variable is quantity). Several types of equilibria appear according to the parameters and the considered variable: either equilibria with a leader and a follower or simultaneous equilibria. In the second configuration, there are cases where being a leader is suboptimal, and both firms wait the second period to move, and by contrast, cases where being a follower is suboptimal, and both firms plays at the first period. Let us focus on the latter case, the most classical one.

In this context, the idea of global $\epsilon$-rationality can be thought in the following way. Even if, at the equilibrium, firms play simultaneously at the first period, one firm may tremble[26] and become a follower. Then, if a firm has several Cournot strategies, it will choose the one that maximizes its payoff taking into account that it might be a leader with probability $\epsilon$. Therefore, global $\epsilon$-rationality can be thought as a trembling-hand refinement, motivated by the ex-post rationality of the trembling agent. The link with local $\epsilon$-rationality appears when the situation is more constrained: assume an incumbent with a given strategy. However, this incumbent fears an entry. Besides, it has another strategy that is strictly better than his current strategy if a potential entrant best responds to this deviation and is equivalent otherwise. It is clear that this deviation can be anticipated by the entrant, making the deviation of the incumbent perfectly rational. It is what Hamilton and Slutsky (1990) may mean when they state:

> Of course, if the existing firms had sufficient postentry flexibility, then entrants will not react to current choices but to their perception of postentry behavior.

This example shows in a simple way how a firm can be incentivized to change its strategy if the entrant's perception about the actual situation is accurate enough. Here, the entrant reacts to the postentry behavior. Therefore, the "observed" strategy is not the reference strategy but the targeted one, since it is what the entrant anticipates. It does correspond to our local $\epsilon$-rationality concept.

## 8.5 Further ideas

Two remarks have to be made. The first one is that among the three notions of rationality we have developed so far, only one leads to an order independent iterated elimination procedure (the proof of this observation is left to the reader but we give some elements of understanding below). How can we explain this lack of consistency?

[26] In the context of a duopoly, the idea of tremble seems quite natural since real life contingencies often delay decision making processes.

With respect to signaling games, self-local rationality seems to be linked to the *intuitive criterion* of Cho and Kreps (1987). Indeed, under self-local rationality, a strategy is not played when it is not a best response if a best response to this strategy is played with probability $\epsilon$. Then, the reference point is the potentially dominated strategy. In the intuitive criterion, the reference point is the tested equilibrium. Broadly, in a signaling game with two types of agents (the senders) and a principal (the receiver), an equilibrium fails the *intuitive test* with respect to a deviation if (*i*) this deviation from the initial equilibrium is never profitable for one type, and if (*ii*) the other type prefers the new equilibrium when the receiver best responds to the deviation. Let us be clear: this criterion might be said global in a sense since we first look at an equilibrium (where everybody best responds) and check if a deviation is profitable (where *only* the receiver best responds). Nevertheless, what interests us in this story is the response of the other type. Indeed, the intuitive criterion *forgets* the optimal reaction of the type for who the deviation is never profitable. That is, the intuitive criterion assumes this type still best responds to the initial equilibrium whereas the deviation leads to another equilibrium. In this sense, the intuitive criterion is self-local. This point had notably been made by Mailath (1988) and led to the notion of *undefeated equilibrium* in Mailath et al. (1993). In fact, this logic is reminiscent of the $E2$ equilibrium in Wilson (1977). Loosely speaking, an equilibrium is said $E2$ if there is no profitable deviation for a player in the following sense: after the opponents' "optimal" reaction[27] to the deviation, the deviation is still profitable, with respect to the initial equilibrium. Since all actions were optimal at the initial equilibrium, and are still optimal when the deviation is tested, we can see the $E2$ equilibrium as a global concept, while the intuitive criterion is well self-local.

In the pure strategy case, global rationality can be stated as follows: player $i$ never wants to play strategy $s_i \in S_i$ once $j$ plays best response to $i$'s strategies with probability $\epsilon$ and $i$ can find another strategy that yields strictly more. However, if the strategy $s_i$ is deemed unplayable, the reason of the elimination may vanish immediately since $i$ requires a best response to $s_i$ to be played[28]. This reasoning similar with self-local rationality. When a player checks whether he should eliminate a strategy, he should not fear losing the payoff if he plays the eliminated strategy, but rather see what he gets if he plays the eliminating strategy. In a word, the situation at the deviation (i.e. by playing the eliminating strategy) should be checked, not the others. In our view, the agent should test a deviation such that this deviation *works*[29] and not

[27]The paper is applied to insurances: thus, the "optimal" reaction is to withdraw insurance policies which reward a negative profit.

[28]This directly shows why an iterated elimination procedure based on global and self-local rationality would be order dependent: these rationality concepts lack hereditariness.

[29]We mean by *works* that the agent gets a strictly higher payoff by deviating rather than playing the reference strategy.

such that the reference point still exists. Indeed, if a deviation is tested and works, it is not necessary anymore to keep in mind the reference point[30]. Finally, when an agent tests a deviation, he *should* anticipate that his opponent will also test it and react accordingly, whatever the true action of the agent.

The second remark is that IERDS is not the only order independent procedure. Additionally, it might not be the only procedure whose dominance is grounded on the existence of a set of profiles which will survive each process (not all the profiles survive but at least one profile does and the set does not expand, letting the required property on the set untouched by each process). In the spirit of RBEU of Cubitt and Sugden (2011), one may think for example that payoff domination on all the profiles where an opponent plays a dominant strategy (if it exists) will survive to any reasonable elimination procedure, and it would be enough. Then, our condition RD2 could be weakened by adding this possibility. This question is still open and might be the object of further research.

# 9 Conclusion

In this paper, we introduce a new dominance relation named *root dominance* between weak and strict dominances. It requires weak dominance and an additional condition based on the *Best Response Set* to the dominating strategy. We associate to this dominance relation an *iterated elimination procedure* named *IERDS*. The main result of this paper is that IERDS is an order independent procedure in finite games and refines IESDS. Additionally, we show that IERDS does not face the inconsistency named *mutability*. Mutability concerns especially IEWDS but also other existing elimination procedure. In a word, mutability means that an eliminated (and thus dominated) strategy in a *process* is finally not dominated at the end of the process. Finally, we introduce new rationality concepts such that our rational strategies correspond to root undominated strategies. Furthermore, we establish a link between our rationality concepts and a *rational* kind of conjectural variations theory, a framework well-known in industrial organization literature and public economics.

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[30] This point is reminiscent of the idea of *memorylessness* developed in Patty (2021).

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# A Omitted proofs

Theorem 2. *IERDS is immutable in finite games.*

*Proof.* Assume there is a strategy $s'_i \in S_i$ eliminated through IERDS. Assume a given process of IERDS and the sequence of games associated $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$. Consider the game formed by $\Gamma^\Lambda$ and the strategy $s'_i$, i.e. the game $\Gamma'$ (with the same players and utilities as $\Gamma^\Lambda$) and the strategy set $S^\Lambda \cup s'_i$. We reason by induction.

*Stage* 1: Assume $s'_i$ has been eliminated by $s''_i$ at step $\Lambda - 1$. Suppose also by contradiction that $s'_i$ is not root dominated in $\Gamma'$. By Lemma 2, $s''_i \in \Gamma^\Lambda \subset \Gamma'$. We repeat the same arguments as in the proof of Lemma 4: comparing $s''_i$ and $s'_i$, it can be verified that RD1 is still respected. By Lemma 3, we know that $\boldsymbol{b}^\Lambda(s''_i) \subseteq \boldsymbol{b}^{\Lambda-1}(s''_i)$. Additionally, $\boldsymbol{b}^\Lambda(s''_i)$ cannot be empty by Lemma 1. Therefore, RD2 is still satisfied and $s'_i$ is not root dominated in $\Gamma'$.

*Stage* $\mu+1$: Now assume the property that *a root dominated strategy* $s'_i$ *at* $\Lambda - \mu$ *is root dominated in* $\Gamma'$ *for a given* $\mu \in [\![2, \Lambda - 1]\!]$ is true. Let us show it is true for $\mu + 1$. Assume the sequence of games $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$ is such that the considered $s'_i$ is eliminated at $\Lambda - (\mu + 1)$.

One can construct a sequence of games $\{\tilde{\Gamma}^\lambda\}_{\lambda \leq \tilde{\Lambda}}$ identical to the previous one until step $\Lambda - (\mu + 1)$, but $s'_i$ is not eliminated at $\Lambda - (\mu + 1)$. By Lemma 5, $s'_i$ is still root dominated in the latter sequence $\{\tilde{\Gamma}^\lambda\}_{\lambda \leq \tilde{\Lambda}}$ at the step $\Lambda - \mu$.

There are two cases: either $(i)$ the strategy $s''_i$ which eliminates $s'_i$ in the first sequence $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$ is in $S^\Lambda (= \tilde{S}^{\tilde{\Lambda}}$ by Theorem 1), and thus is never eliminated; or $(ii)$ the strategy $s''_i$ is eliminated at a further step of the sequence $\{\Gamma^\lambda\}_{\lambda \leq \Lambda}$.

In the former case $(i)$, it is straightforward to show that $s'_i$ is root dominated in $\Gamma'$ by repeating the arguments used in the first stage of our induction reasoning.

In the latter case $(ii)$, we know by the induction hypothesis that $s''_i$ is root dominated in the game $\Gamma''$ (where $\Gamma''$ is analogous to $\Gamma'$ with the same players and utilities as $\Gamma^\Lambda$ and the strategy set $S^\Lambda \cup s''_i$). Clearly, if $s''_i$ is root dominated by a strategy $s'''_i$

in $\Gamma''$, since $s_i''$ always very weakly dominates $s_i'$, $s_i'$ is also root dominated by $s_i'''$ in $\Gamma'$ (since $\Gamma'$ and $\Gamma''$ are the same games but either $s_i'$ or $s_i''$ is added to $\Gamma^\Lambda = \tilde{\Gamma}^{\tilde{\Lambda}}$).

*Conclusion*: We have shown that a strategy eliminated at $\Lambda - (\mu + 1)$ is still virtually root dominated at the end of the sequence. Thus, by induction, it is true for each $\mu \in [\![2, \Lambda]\!]$. Since we did not need any assumption on the process used to construct our initial sequence, this result is true for any process. ■

# B Additional results

The next result states that no finite game becomes empty through IERDS:

**Fact 2.** *$S^0 \neq \emptyset \Rightarrow \forall \{\Gamma^\lambda\}_{\lambda \leq \Lambda}$, $S^\Lambda \neq \emptyset$.*

*Proof.* By Proposition 1 and by the finiteness of the games, it is clear that for each strategy set, there is at least one undominated strategy that can never been eliminated. ■

IERDS satisifies the Individual Independence of the Irrelevant Alternatives (IIIA) as formulated by Gilboa et al. (1990), i.e. the addition of one $i$'s strategy does not affect the dominance relation between $i$'s strategies:

**Proposition 3.** *Assume $\Gamma$ and $\Gamma'$ two games such that $N = N', S_{-i} = S'_{-i}, U = U'$, and $S'_i = S_i \cup s_i^*$. Then:*

$$s_i'' \succ s_i' \text{ in } \Gamma \Rightarrow s_i'' \succ s_i' \text{ in } \Gamma'.$$

*Proof.* Adding $s_i^*$ does not affect the payoff of $i$ when playing $s_i'$ and $s_i''$. As well it does not affect the profiles in $\boldsymbol{b}(s_i'')$. Thus, if all conditions of Definition 4 are checked in $\Gamma$, it is also the case in $\Gamma'$. ■

Nevertheless, we cannot use the main result of Gilboa et al. (1990) that states the order independence of *hereditary* dominance relations which are partial orders and respect IIIA. Indeed, root dominance is not *hereditary* in their sense:

**Definition 19.** *Assume $\Gamma$ and $\Gamma'$ such that $N = N', U = U', S' \subset S$. If $S \subset S'$, then the well defined dominance relation $\succ$ is said hereditary if:*

$$\forall s_i'', s_i' \in S_i,\ s_i'' \succ s_i' \text{ in } \Gamma' \Rightarrow s_i'' \succ s_i' \text{ in } \Gamma'.$$

The next example shows why $\succ$ is not hereditary. We show in green the Nash equilibria. In the game below, IERDS eliminates $B$, then $K1$ and finally $L$:

$k$'s Strategy

| | $j$'s Strategy | | | $j$'s Strategy | |
|---|---|---|---|---|---|
| $i$ | $L$ | $R$ | $i$ | $L$ | $R$ |
| $T$ | $(3,1,1)$ | $(3,0,1)$ | $T$ | $(3,0,2)$ | $(3,1,2)$ |
| $B$ | $(2,1,1)$ | $(3,0,1)$ | $B$ | $(2,0,0)$ | $(2,0,0)$ |
| | $K1$ | | | $K2$ | |

$\longrightarrow$

$k$

| | $j$ |
|---|---|
| $i$ | $R$ |
| $T$ | $(3,1,2)$ |
| | $K2$ |

Table 3: Games where hereditariness fails

However if we arbitrarily suppress the strategy $L$ of the game, then no elimination can be made with IETDS. Therefore, root dominance is not hereditary, since $B$ is not root dominated by $T$ in the following "subgame":

$k$'s Strategy

| | $j$ | | $j$ |
|---|---|---|---|
| $i$ | $R$ | $i$ | $R$ |
| $T$ | $(3,0,1)$ | $T$ | $(3,1,2)$ |
| $B$ | $(3,0,1)$ | $B$ | $(2,0,0)$ |
| | $K1$ | | $K2$ |

Table 4: Subset of the previous Game with no Possible Elimination

# C Best reply set

Now we show with the next example (we show in green the Nash equilibria) why the definition of the Best Reply Set requires to consider all profiles where (at least) one opponent best responds and not only where *all* the opponents mutually best respond (beyond the obvious problem of existence with more than two players):

$k$'s Strategy

| | $j$'s Strategy | | |
|---|---|---|---|
| $i$ | $L$ | $C$ | $R$ |
| $T$ | $(3,3,1)$ | $(3,3,1)$ | $(3,2,1)$ |
| $M$ | $(3,2,1)$ | $(3,1,0)$ | $(3,1,0)$ |
| $B$ | $(3,2,1)$ | $(3,1,0)$ | $(3,1,2)$ |

$K1$

| | $j$'s Strategy | | |
|---|---|---|---|
| $i$ | $L$ | $C$ | $R$ |
| $T$ | $(3,3,1)$ | $(3,3,1)$ | $(3,2,2)$ |
| $M$ | $(3,0,2)$ | $(3,2,0)$ | $(4,2,1)$ |
| $B$ | $(3,0,2)$ | $(3,2,0)$ | $(3,2,1)$ |

$K2$

| | $j$'s Strategy | | |
|---|---|---|---|
| $i$ | $L$ | $C$ | $R$ |
| $T$ | $(3,3,1)$ | $(3,3,1)$ | $(3,2,1)$ |
| $M$ | $(3,2,0)$ | $(3,1,2)$ | $(3,0,0)$ |
| $B$ | $(3,2,0)$ | $(3,1,2)$ | $(3,0,0)$ |

$K3$

Table 5: Best Reply Set with a Three Players Game

In this game, if the Best Reply Set definition was modified, $M$ would root dominate $B$ and $T$. Indeed, the only profile where $j$ and $k$ mutually best respond to $M$ is $(M, R, K2)$. At this profile $M$ is strictly better than the two other strategies. Since $M$ weakly dominates the two other strategies, it would be done. However, it can be easily seen that $C$ would also root dominate $R$ (all mutual best responses to $C$ are indeed parts of the Nash equilibria). Then the order of elimination would matter. What is important here is that at $(M, C, K2)$, $k$ strictly wants to deviate, making the profile unchecked with a modified version of the Best Reply Set.

## D Are inadmissible strategies playable?

Despite the inconsistencies of IEWDS, one may still assert that weakly dominated strategies *should* not be played. For instance, Kohlberg and Mertens (1986, p. 1014) justify admissibility as a criterion of strategic stability with the following reasoning: assume a pure strategy two-player game with player $i$ having one strategy $s_i''$ which weakly dominates $s_i'$ and additionally, such that if $i$ is indifferent between $s_i''$ and $s_i'$, $j$ is also indifferent at these profiles (it is the TDI condition of Marx and Swinkels (1997)). Now, the game has the next extensive form (see Figure 9): first, $i$ is asked to choose between $(s_i', s_i'')$ and all of his other strategies. Second, $j$ chooses his strategy. Finally, there is a third stage only if $i$ has chosen $(s_i', s_i'')$ at the first step and if $s_i'$ and $s_i''$ do not give the same payoffs (i.e. if $j$ has chosen a strategy among the strategies $\succnsim$ where both players are not indifferent with respect to the choice of $i$ between $s_i'$ and $s_i''$). Kohlberg and Mertens (1986) claim that in this form of game, $s_i'$ is never played. It is true. However, Kohlberg and Mertens (1986) do not consider the games with payoffs such that $s_i''$ is never played either. Let us see the behavior of $j$ if $i$ has chosen

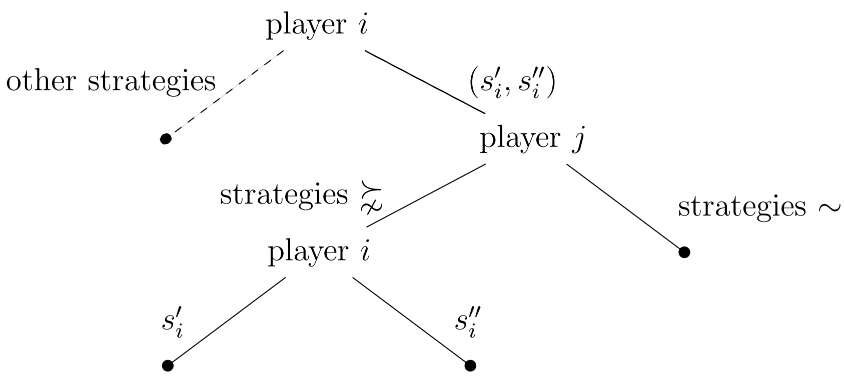


Figure 9: Extensive-Form Game: $s_i''$ Weakly Dominates $s_i'$ but might Never be Played in the third stage

the couple $(s_i', s_i'')$ rather than another strategy at the first stage. Then, $j$ necessarily plays a best response to $s_i'$ or $s_i''$. Assume $j$ chooses a best response $s_j^*$ to $s_i'$. Then, either $i$ is not indifferent and will necessarily choose $s_i''$ (since $s_i''$ weakly dominates $s_i'$, if $i$ is not indifferent, he strictly prefers $s_i''$), making the choice of $j$ suboptimal if it is not a best response to $s_i''$ too, or $i$ is indifferent. In this latter case, by assumption (the TDI condition), $j$ is also indifferent. However, if $s_j^*$ is not a best response to $s_i''$, then $j$ could have obtained a strictly higher payoff by deviating towards a best response to $s_i''$. Therefore, in this part of the game, $j$ always plays a best response to $s_i''$. Then, $i$ may play $s_i''$ at the third stage only if players are not indifferent at (at least) one profile where $j$ best responds to $s_i''$. If there is (at least) one best response of $j$ to $s_i''$ such that $i$ and $j$ are indifferent between $s_i'$ and $s_i''$, then $s_i''$ might be never played. The idea of Kohlberg and Mertens (1986) is that a strategy is inadmissible if it is never played in such an extensive-form game. Nevertheless, this criterion cannot characterize inadmissibility since an admissible strategy might never be played either (if $j$ always plays a strategy in $\sim$), according to the considered game. We claim that one possibility is to choose a more cautious criterion: $s_i'$ is dominated by $s_i''$ if and only if $s_i''$ is always played in this part of the game. Precisely, we should require that $s_i''$ is played with probability 1 in the third stage when $i$ chooses the couple $(s_i', s_i'')$ at the first stage. In this case, $s_i''$ should strictly payoff dominate $s_i'$ where $j$ best responds to $s_i''$. It is exactly our second condition of dominance. Note that the reasoning we have just made does require weak dominance, like our notion of dominance does.

Besides, remark that root dominance differs from the notion of nice weak dominance introduced by Marx and Swinkels (1997) since nice weak dominance is equivalent to weak dominance in games where the TDI condition is respected. Thus, in all the games we have considered, $s_i'$ is *nicely weakly dominated* by $s_i''$. One can see why the iterated elimination of nicely weakly dominated strategies is payoff order independent in such games with the two following examples:

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(\underline{2},\underline{2})$ | $(\underline{3},1)$ |
| $B$ | $(\underline{2},\underline{2})$ | $(2,1)$ |

$\xrightarrow{\text{IEWDS}}$

| | $j$'s Strat. |
|---|---|
| $i$'s Strat. | $L$ |
| $T$ | $(\underline{2},\underline{2})$ |
| $B$ | $(\underline{2},\underline{2})$ |

or

| | $j$'s Strat. |
|---|---|
| $i$'s Strat. | $L$ |
| $T$ | $(\underline{2},\underline{2})$ |

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(\underline{2},2)$ | $(\underline{3},\underline{3})$ |
| $B$ | $(\underline{2},2)$ | $(2,\underline{3})$ |

$\xrightarrow{\text{IEWDS}}$

| | $j$'s Strat. |
|---|---|
| $i$'s Strat. | $R$ |
| $T$ | $(\underline{3},\underline{3})$ |

Figure 10: IEWDS applied to Games with the TDI Condition

In the top game, $j$ best responds to $T$ by playing $L$. At this profile, $i$ is indifferent between $T$ and $B$. With respect to our previous remarks, it might be problematic. Indeed, here, the order of deletion of IEWDS matters: the outcome of IEWDS is either $(T, L)$ or $(\Delta(T, B), L)$. Nevertheless, thanks to the TDI condition, it does not affect the payoffs. Again, in this paper, we consider such an outcome of IEWDS as an example of order dependence. Now, in the bottom game, $j$ best responds to $T$ by playing $R$. There, $i$ is not indifferent, and the order does not matter, the outcome of IEWDS always being $(T, R)$. One can remark that the TDI condition does not matter either in this game. Indeed, whatever the payoff of $j$ at the profile $(B, L)$, IEWDS would still be order independent. Naturally, we depart from the notion of nice weak dominance since root dominance requires payoff dominance at the profiles where $j$ best responds to $T$.

# E Weak dominance in $2 \times 2$ games

Assume the following general form for a $2 \times 2$ game where $T$ weakly dominates $B$ (i.e. $a > c$):

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(a, \alpha)$ | $(a, \beta)$ |
| $B$ | $(c, \gamma)$ | $(a, \delta)$ |

Table 6: Game with Weak Dominance

With respect to the value of $\alpha, \beta, \gamma$ and $\delta$, there are 9 possible configurations that we gather in subsets according to their properties. The three configurations $(i)$ where $\alpha > \beta$ are order independent and immutable. The special configuration $(i')$ where $\alpha = \beta$ and $\gamma = \delta$ is order independent and immutable as well. The configurations $(ii)$ with $\alpha < \beta$ and $\gamma > \delta$ is order independent but is mutable[31]. Finally, other configurations $(iii)$ are order dependent and mutable (those either with $\alpha = \beta$ and $\gamma \neq \delta$ or $\alpha < \beta$ and $\gamma \leq \delta$). Configurations $(i)$ correspond to cases where $T$ root dominates $B$. All other configurations are such that $(T, R) \in \boldsymbol{b}(T)$, and therefore $T$ does not root dominate $B$. Note that $(ii)$ differs from $(iii)$ also because $j$ does not have (weakly) dominated strategy in $(ii)$. In this game, except for configuration $(i')$, both root dominance solvability and consistency of IEWDS correspond to cases where the selected Nash equilibrium is strict (but not necessarily Pareto-dominant), i.e. no player has a payoff-equivalent unilateral deviation. Other cases are such that no Nash equilibrium is strict. The case $(i')$ is such that IERDS eliminates no strategy. In contrast, IEWDS eliminates $B$ and that is all. Again, the configuration $(i')$ is special. However, it shows that IERDS fails to delete some strategies which are virtually dominated in the IEWDS outcome. Thus, IERDS is not the "maximal" immutable elimination procedure.

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(a, \alpha)$ | $(a, \alpha)$ |
| $B$ | $(c, \alpha)$ | $(a, \alpha)$ |

Table 7: Game with configuration $(i')$: IEWDS eliminates $B$

[31]Note that it can be compared to Samuelson (1992, Example 8) which shows that common knowledge of admissibility may not exist. Note that if in addition to these specifications, we assume that $\beta = \delta$, this game respects the *transference of decisionmaker indifference (TDI)* condition of Marx and Swinkels (1997) which ensures the *outcome* order independence of IEWDS in finite games (i.e. any order of elimination leads to the same payoffs). Therefore, it shows that *nice weak dominance* (which is equivalent to weak dominance in the class of finite TDI games) may exhibit mutability.

# F Pareto dominance and pre-play communication

Note that our procedure does not lead to the selection of the Pareto dominant equilibrium (we show in green the Nash equilibria). Even if the Pareto dominant strictly dominates another equilibrium, the latter may still be selected instead as it is shown with this example:

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(2, 1)$ | $(1, 2)$ |
| $B$ | $(2, 3)$ | $(0, 0)$ |

Table 8: Game with a Pareto Dominated Unique Prediction

However, if we define a strict Nash equilibrium as an equilibrium where each player best responds to the profile and this best response is unique (i.e. no strategy is payoff equivalent at this profile), we can easily show that IERDS never eliminates this kind of equilibrium:

**Fact 3.** *IERDS does not eliminate strict Nash equilibria.*

The proof is immediate since if a profile is a strict Nash equilibrium, then all strategies of the profile cannot be iteratively weakly dominated. Note that IEWDS does not eliminate strict Nash equilibria by the same argument.

Remark that if we invoke the notion of *self signaling*[32], $(T, R)$ is the only equilibrium such that both agents play a self signaling action. Briefly, in a two-players game with pre-play communication, an action is said *self signaling* if the action the sender announces is a strict best response if his opponent plays a best response to this action; if he plays another action, he strictly prefers that the opponent plays another strategy. Therefore there should not be an incentive to deviate for the sender once he thinks his opponent trusts him[33]. It is not surprising that a root dominating strategy enables a

[32] *Self signaling* is described in Farrell and Rabin (1996) for pre-play communication, see Baliga and Morris (2002) for formal definitions

[33] In other words, if the sender announces something that he always wants to be believed (whatever it is true or false), his commitment is weak, he cannot self signal. Here, conditions to have $T$ and $R$ self signaling are: $U_i(T, R) > U_i(B, R)$ and $U_i(B, R) < U_i(B, L)$ for agent $i$ (conditionally to $j$ best responding at $(T, R)$) and symmetrically $U_j(T, R) > U_j(T, L)$ and $U_j(T, L) < U_j(B, L)$ for agent $i$ (conditionally to $i$ best responding at $(T, R)$).

strongly believed commitment since it is an undominated strategy[34]. At $(B, L)$, only $j$ can self signal, while $i$ cannot even *self commit* (*self commitment* requires only that the action announced is a strict best response if the opponent plays a best response). Moreover, even if $(B, L)$ is the Pareto dominant profile, action $L$ is not a *Stackelberg* action (i.e. the unique preferred action if the opponent always plays a best response) because $T$ is also a best response to $L$, and at $(T, L)$, $j$ wants to deviate.

# G Proof of mixed IERDS order independence result

First, it is obvious that Lemma 1 still applies. Now, we state that mixed root dominance forms also a strict partial order:

**Proposition 4.** *Mixed root dominance is a strict partial order: it is a binary relation such that irreflexivity, asymmetry and transitivity hold.*

The proof is analogous to the pure strategy case:

*Proof.* Root dominance is irreflexive: by Lemma 1, $\forall \sigma_i, \in \Sigma_i$ $\boldsymbol{b}(R_{\sigma_i}) \neq \emptyset$, and it is not possible to have $U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i, \sigma_{-i})$ for any profile $\sigma_{-i} \in \Sigma_{-i}$. Then, RD2$'$ cannot be respected. Root dominance is transitive: assume $\sigma_i'' \succ \sigma_i'$ and $\sigma_i''' \succ \sigma_i''$. Here, we have to prove that $\sigma_i''' \succ \sigma_i'$. First, it is straightforward that RD1$'$ is respected. Second, since $\sigma_i''' \succ \sigma_i''$, we know that $U_i(\sigma_i''', \sigma_{-i}) > U_i(\sigma_i'', \sigma_{-i})$ for each strategy profile $\sigma_{-i}$ contained in $\boldsymbol{b}(R_{\sigma_i'''})$. Since $\sigma_i'' \succ \sigma_i'$, $U_i(\sigma_i'', \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$ for each strategy profile $\sigma_{-i}$ in $\Sigma_{-i}$, and thus for each strategy profile $\sigma_{-i}$ contained in $\boldsymbol{b}(R_{\sigma_i'''})$. Therefore, $U_i(\sigma_i''', \sigma_{-i}) > U_i(\sigma_i'', \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$ for each strategy profile $\sigma_{-i}$ contained in $\boldsymbol{b}(R_{\sigma_i'''})$ and RD2$'$ is respected. Finally, irreflexivity and transitivity together imply asymmetry. ■

**Lemma 9.** *If $s_i' \in S_i$ is root dominated, there is (at least) one strategy $\sigma_i'' \in \Sigma_i$ that may eliminate it, i.e. a strategy $\sigma_i''$ whose no strategy in the support is root dominated by an other strategy and root dominates $s_i'$.*

*Proof.* Since the number of pure strategies is finite, the number of pure strategies contained in all the supports containing (mixed) strategies root dominating $s_i'$ is necessarily finite. Let us denote it $m$ and denote $g(s_i')$ the set of these strategies. Then,

[34] Note that if we modified the payoff of $i$ such that $i$ earns $-1$ when $j$ plays $L$, there would still be root dominance but $T$ would not be self signaling. Then, there is obviously not equivalence between the two concepts.

(at most) $m-1$ of these strategies are root dominated. Otherwise, it means that the $m^{th}$ strategy, named $s_i''$, is root dominated by an other strategy whose support contains (at least) one pure strategy outside $g(s_i')$. By transitivity of root dominance, it means that the latter strategy also root dominates $s_i'$, contradicting the fact that the number of pure strategies contained in all the supports containing (mixed) strategies root dominating $s_i'$ is $m$. Thus, we have established that at least $s_i''$ is not root dominated. Additionally, by Lemma 7, we know that the $m-1$ strategies root dominated are root dominated by strategies $\tilde{\sigma}_i \in \Sigma_i$ whose supports do not contain them. Again, by transitivity, the support of these strategies $\tilde{\sigma}_i$ is necessarily contained in $g(s_i')$. Therefore, $s_i''$ root dominates each of these strategies: otherwise, either one of these strategies is not root dominated and there is a contradiction, or it is dominated by a strategy whose support is outside $g(s_i')$, a contradiction. Finally, $s_i''$ root dominates $s_i'$. We can make the same reasoning when more than one pure strategy is not root dominated and the proof is done.

■

It is straightforward to show that $\boldsymbol{b}(R_{\sigma_i})$ never expands as we progress through the steps of mixed IERDS thanks to the previous result. Finally, all remaining results are written in the same until the hereditariness result, and we get the order independence result.

# H Burned money

Root dominance fails to be as predictive as IEWDS or Iterated Elimination of Choice sets under Full Admissible consistency (IECFA) of Asheim and Dufwenberg (2003) if we study the battle of sexes game with a burning option for one player (see for instance Rubinstein (1991, p.920)). If payoffs are as described in the payoffs matrix of Table 9 (we show in green the Nash equilibria), we only delete one strategy for each agent, eliminating one Nash equilibrium. This result is not completely satisfying since we preserve a strategy where money is burnt and the equilibrium deleted is the one where the second agent has the maximal payoff. $(BD)$ is root dominated by $(NU)$. This deletion is necessary to eliminate $(RR)$ (by $(RL)$) but no further elimination is possible.

| | $j$'s Strategy | | | |
|---|---|---|---|---|
| $i$'s Strat. | $LL$ | $LR$ | $RL$ | $RR$ |
| $NU$ | $(3,1)$ | $(3,1)$ | $(0,0)$ | $(0,0)$ |
| $ND$ | $(0,0)$ | $(0,0)$ | $(1,3)$ | $(1,3)$ |
| $BU$ | $(2,1)$ | $(-1,0)$ | $(2,1)$ | $(-1,0)$ |
| $BD$ | $(-1,0)$ | $(0,3)$ | $(-1,0)$ | $(0,3)$ |

$\xrightarrow[\text{pure IERDS}]{}$

| | $j$'s Strategy | | |
|---|---|---|---|
| $i$'s Strat. | $LL$ | $LR$ | $RL$ |
| $NU$ | $(3,1)$ | $(3,1)$ | $(0,0)$ |
| $ND$ | $(0,0)$ | $(0,0)$ | $(1,3)$ |
| $BU$ | $(2,1)$ | $(-1,0)$ | $(2,1)$ |

Table 9: Burned Money in Rubinstein (1991)

However, if we allow a mixed extension of the game, mixed strategies where $BU$ is more used than $NU$ root dominate $ND$. Then $LL$ root dominates $RL$. Finally we end the procedure by eliminating $BU$, and we get the two Nash equilibria favoring $i$:

$\xrightarrow[\text{mixed IERDS}]{}$

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strat. | $LL$ | $LR$ |
| $NU$ | $(3,1)$ | $(3,1)$ |

Table 10: Final outcome of Burned Money after mixed IETDS

# I Additional concepts characterizing root dominance

In this part, we introduce two additional rationality concepts which can characterize root dominance. For this purpose, we introduce two new types of games where the perception of player $j$ is perturbed with probability $\epsilon$.

In the first configuration, the *hesitation game*, we suppose that despite having a "reference" strategy (unobserved by the opponent), a player may alternatively consider some strategy subsets. If so, the opponent reacts optimally (in a naive way) to this strategy subset. Thus, the reference strategy is "tested" against such mind trembles. If the reference strategy is not optimal when he believes that the opponent can detect this tremble and react optimally, the tremble should be realized and, in fact the reference strategy never played in such a game.

In the second configuration, named *deviation game*, the opponent observes both the reference strategy and the strategy subset from which a potential deviation is picked.

Closer to the spirit of Fudenberg et al. (1988), $j$ analyses whether the deviation is sustainable before reacting optimally. That is, we assume $j$ plays a best response to the deviation if and only if it the deviation is deemed *credible* with respect to the reference strategy. We summarize $j$'s beliefs in Figure 11.

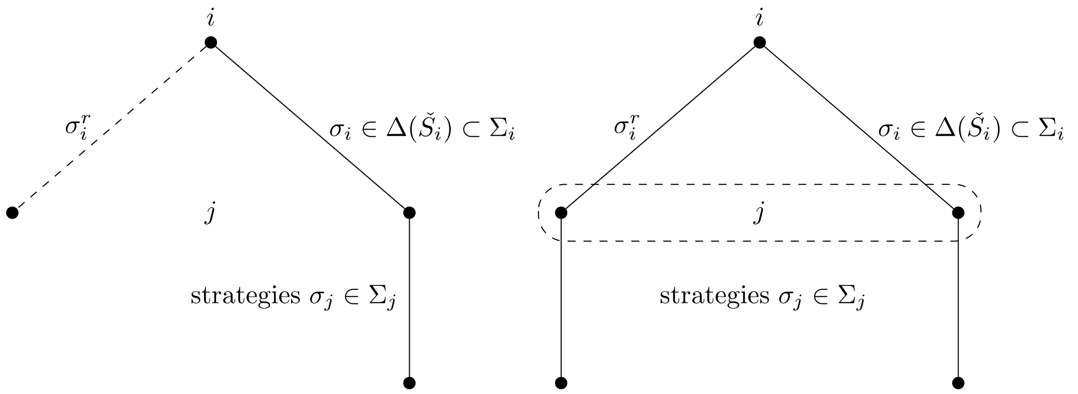


Figure 11: Beliefs of $j$ if a Perturbation occurs in Hesitation (left) and Deviation (right) Games

## I.1 Hesitation games

Assume each player believes that with probability $\epsilon$ he may "hesitate". That is, if he has chosen a reference strategy $\sigma_i^r \in \Sigma_i$, he may think to choose other (mixed) strategies supported by any strategy subset $\check{S}_i \subset S_i$. Additionally, assume that this thought is observable by the opponent $j$ and that $j$ believes that $i$ will actually play a strategy in $\Delta(\check{S}_i)$. At this point, $i$ can substitute a strategy in $\Delta(\check{S}_i)$ for $\sigma_i^r$. If with such a perturbation, $\sigma_i^r$ does not maximize $i$'s utility, then $\sigma_i^r$ should not be played.

First, we define two concepts which assume restrictions on the available strategies:

**Definition 20.** *A restricted game $\widehat{\Gamma}(\sigma_i^r, \check{S}_i)$ is a simultaneous game such that player $i$ chooses a strategy $\sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i)$ where $\check{S}_i \subset S_i$, and such that it is common knowledge that player $j$ believes with probability 1 that $\Sigma_i = \Delta(\check{S}_i)$.*

A *restricted game* $\widehat{\Gamma}(\sigma_i^r, \check{S}_i)$ is a game where the strategy set is $\sigma_i^r \cup \Delta(\check{S}_i) \times S_j$ but player $j$ believes that the strategy set is $\Delta(\check{S}_i) \times \Sigma_j$. Now, we can define the *$\epsilon$-hesitation game*, whose name indicates that players might *hesitate* with probability $\epsilon$:

**Definition 21.** *An $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$ for player $i$ and strategy $\sigma_i^r$ is a game where:*

1. *Player $i$ chooses the strategy $\sigma_i \in \Sigma_i$, and player $j$ chooses a strategy in $\Sigma_j$ with probability $1 - \epsilon$,*

*2. With probability $\epsilon$, $i$ and $j$ play a restricted game $\widehat{\Gamma}(\sigma_i^r, \check{S}_i)$.*

That is, we assume the perception of the game by player $j$ is restricted to $\Delta(\check{S}_i) \times \Sigma_j$ with probability $\epsilon$. Clearly, the perception can be false since $i$ is allowed to choose the strategy $\sigma_i^r$. However, we assume that $j$ almost guesses the thought of $i$ with probability $\epsilon$, since $j$ perceives (at least) partially where the attention of $i$ is.

Furthermore, we can make a link with the idea of deviation and its observation. Indeed, an $\epsilon$-hesitation game formalizes the reasoning process of player $i$ when:

1. The "usual" strategy of player $i$ is $\sigma_i^r$,
2. Player $i$ thinks about a deviation to any other strategy contained in $\Delta(\check{S}_i)$,
3. Opponent $j$ observes with probability $\epsilon$ that $i$ is thinking to choose a strategy in $\Delta(\check{S}_i)$.

The consequence of step 3 is that $i$ believes that $j$ will choose a best response to $\Delta(\check{S}_i)$ with probability $\epsilon$.

More concretely, the reasoning is the following. When $i$ thinks about whether a strategy $\sigma_i^r$ is "playable", he takes it as a reference point. Then, he wonders whether he may want to deviate. For this purpose, he considers all strategy subsets $\check{S}_i$. For each one, he believes that $j$ will react optimally with probability $\epsilon$. Finally, he checks if he would want to deviate from $\sigma_i^r$ in all cases verifying this belief. If there is a deviation that yields strictly more, player $i$ never chooses $\sigma_i^r$ to avoid to pay the cost $c$ when facing the restricted game. One could remark that the behavior of $j$ seems too "naive". In the next subsection we introduce a second kind of perturbation that tackles this issue.

Now, we define the best response of an $\epsilon$-hesitation game:

**Definition 22.** *Consider an $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$. A strategy $\sigma_i^* \in \sigma_i^r \cup \Delta(\check{S}_i)$ is a best response of the $\epsilon$-hesitation game if:*

$$\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \boldsymbol{b}(\check{S}_i), \forall \sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i),\ V_i^\epsilon(\sigma_i^*, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \tag{H-BR}$$

Finally, we introduce the concept of $\epsilon$-hesitation dominance which formalizes the dominance relation when we consider the expected $\epsilon$-perturbed utility, and such that the dominating strategy is "observed" by the opponent:

**Definition 23.** *A strategy $s_i \in S_i$ is $\epsilon$-hesitation dominated by $\sigma_i \in \Sigma_i$ if:*

$$\forall \sigma_j \in \Sigma_j, \forall \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i}),$$
$$V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*)$$

In words, $\sigma_i$ $\epsilon$-hesitation dominates $s_i$ when $j$ reacts optimally to $\sigma_i$ with probability $\epsilon$. Naturally, this concept seems similar to root dominance. Indeed, Lemma 12 shows their equivalence when $\epsilon \to 0^+$.

**Lemma 10.** *A strategy $s_i \in S_i$ is $\epsilon$-hesitation dominated if and only if it is a never best response of (at least) one $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$.*

*Proof.* Assume a strategy $s_i$ is $\epsilon$-hesitation dominated by $\sigma_i \in \Sigma_i$. It means that if $R_{\sigma_i}$ is observed with probability $\epsilon$, then the utility from playing $\sigma_i$ is strictly higher than from playing $s_i$. Therefore, $s_i$ is never best response of the $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(s_i, R_{\sigma_i})$. Now, by contrapositive, assume that $s_i$ is not $\epsilon$-hesitation dominated by any $\sigma_i \in \Sigma_i$ and let us show it is a best response to a belief for $i$ when $i$ plays a given $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$. Consider the vectors $\overrightarrow{V_i^\epsilon}(\sigma_i, \check{S}_i) = \{V_i^\epsilon(\sigma_i, s_j, s_j^*)\}_{s_j \in S_j, s_j^* \subset \boldsymbol{b}(\check{S}_i)}$ for each $\sigma_i \in s_i \cup \Delta(\check{S}_i)$. Simply, these vectors are such that each component $l + m$ is the payoff $i$ can obtain when playing $\sigma_i$ and when $j$ plays the pure strategy $s_j^l \in S_j$ with probability $1 - \epsilon$ and the pure strategy $s_j^{*^m} \subset \boldsymbol{b}(\check{S}_i)$ with probability $\epsilon$. We denote $Y(s_i, \check{S}_i)$ the set of such vectors. Besides, we can construct the following set $X$. If $k$ is equal to $\sharp(S_j) \times \sharp(\boldsymbol{b}(\check{S}_i))$[35], then $X$ is the set $\left\{x \in \mathbb{R}^k \,|x > \overrightarrow{V_i^\epsilon}(s_i)\right\}$, that is the set of all payoffs that strictly dominate $s_i$ payoffs. Both $X$ and $Y(s_i, \check{S}_i)$ are convex sets. Since $s_i$ is not $\epsilon$-hesitation dominated, these sets are disjoint. Then, we can apply the separating hyperplane theorem which states that there is a vector in $\mathbb{R}^k$, $\pi \geq 0$ with $\pi \neq 0$ and such that:

$$\forall y \in Y(s_i, \check{S}_i), \forall x \in X, \pi \cdot y \leq \pi \cdot \overrightarrow{V_i^\epsilon}(s_i) \leq \pi \cdot x$$

It directly implies that $\forall \sigma_i \in s_i \cup \Delta(\check{S}_i),\ \pi \cdot \left(\overrightarrow{V_i^\epsilon}(s_i) - \overrightarrow{V_i^\epsilon}(\sigma_i)\right) \geq 0$.

Now, remark that this is true for every hesitation game and finally we get the result.

■

Conversely, a strategy $s_i \in S_i$ being $\epsilon$-hesitation undominated is a best response in all $\epsilon$-hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$. Though, it does not mean that $s_i$ necessarily verifies Equation (1):

$$\exists \sigma_j \in \Sigma_j, \text{ such that } \forall \sigma_i \in \Sigma_i, \exists \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i}), \quad V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*) \qquad (1)$$

Here, we stress the fact that the strategy $\sigma_j \in \Sigma_j$ is not necessarily the same for all the hesitation games when it is stated that a strategy is a best response in all hesitation games. The equivalence holds only when $\epsilon \to 0^+$:

[35] We denote $\sharp(S_i)$ the number of elements in the set $S_i$.

**Lemma 11.** *A strategy $s_i \in S_i$ is a best response to all $\epsilon$-hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$ when $\epsilon \to 0^+$ if and only it verifies Equation (1) when $\epsilon \to 0^+$.*

*Proof.* Set $\epsilon \to 0^+$. Assume that $s_i$ is a best response in all $\epsilon$-hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$. Then,

$$\forall \sigma_i \in \Sigma_i, \exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i}), \text{ such that } V_i^\epsilon(s_i, \sigma_j, \sigma_j^*) \geq V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*)$$

By continuity of $V_i^\epsilon$ in parameter $\epsilon$, it is immediate that we have:

$$\forall \sigma_i \in \Sigma_i, \exists \sigma_j \in \Sigma_j \text{ such that } \mathbb{E}[U_i(s_i, \sigma_j)] \geq \mathbb{E}[U_i(\sigma_i, \sigma_j)]$$

By Pearce (1984, Lemma 3), the previous equation is equivalent to:

$$\exists \sigma_j \in \Sigma_j \text{ such that } \forall \sigma_i \in \Sigma_i, \; \mathbb{E}[U_i(s_i, \sigma_j)] \geq \mathbb{E}[U_i(\sigma_i, \sigma_j)]$$

This last equation is well equivalent to Equation (1) when $\epsilon \to 0^+$. The same reasoning as above can be applied to show the converse part of this result. ∎

Now, we state the equivalence between hesitation dominance when the perturbation occurs with an infinitesimal probability and root dominance:

**Lemma 12.** *A strategy is $\epsilon$-hesitation dominated when $\epsilon \to 0^+$ if and only if it is root dominated.*

*Proof.* The "if" part is straightforward. Indeed, assume that $s_i \in S_i$ is root dominated by $\sigma_i \in \Sigma_i$. First, RD1′ and RD2′ imply that $\sigma_i$ weakly dominates $s_i$. Thus, $\forall \sigma_j \in \Sigma_j$, $U_i(\sigma_i, \sigma_j) \geq U_i(s_i, \sigma_j)$. Second, RD2′ states that for each best response to a strategy in the support of $\sigma_i$, the expected payoff from playing $\sigma_i$ is strictly higher. Therefore, $\forall \sigma_j^*(\sigma_i) \subset \boldsymbol{b}(R_{\sigma_i})$, we have $U_i(\sigma_i, \sigma_j^*(\sigma_i)) > U_i(s_i, \sigma_j^*(\sigma_i))$. Then, for any $\epsilon > 0$, and $\forall \sigma_j \in \Sigma_j, \forall \sigma_j^*(\sigma_i) \subset \boldsymbol{b}(R_{\sigma_i})$:

$$V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*(\sigma_i)) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*(\sigma_i))$$

For the "only if" part, assume that $s_i$ is $\epsilon$-hesitation dominated by $\sigma_i$ but root undominated by $\sigma_i$. Undomination means that either $(i)$ there is a $\sigma_j \in \Sigma_j$ such that $\mathbb{E}[U_i(s_i, \sigma_j)] > \mathbb{E}[U_i(\sigma_i, \sigma_j)]$ or $(ii)$ there is a $\sigma_j^{**} \subset \boldsymbol{b}(R_{\sigma_i})$ such that $\mathbb{E}[U_i(s_i, \sigma_j^{**})] \geq \mathbb{E}[U_i(\sigma_i, \sigma_j^{**})]$. About $(i)$, we remark that $V_i^\epsilon$ is continuous in the parameter $\epsilon$. Then, it is not possible to have simultaneously $\mathbb{E}[U_i(s_i, \sigma_j)] > \mathbb{E}[U_i(\sigma_i, \sigma_j)]$ and $\forall \sigma_j^*(\sigma_i) \subset \boldsymbol{b}(R_{\sigma_i})$, $V_i^\epsilon(\sigma_i, \sigma_j, \sigma_j^*(\sigma_i)) > V_i^\epsilon(s_i, \sigma_j, \sigma_j^*(\sigma_i))$ when $\epsilon \to 0^+$. Besides, the hypothesis $(ii)$ directly implies that $V_i^\epsilon(s_i, \sigma_j^{**}, \sigma_j^{**}) \geq V_i^\epsilon(\sigma_i, \sigma_j^{**}, \sigma_j^{**})$. In both cases, there is a contradiction with the hypothesis of $0^+$-perturbed dominance. ∎

Finally, we can state the first main result of this section, namely the equivalence between root dominance of $s_i$ and rationality when considering all the $\epsilon$-hesitation games $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$ associated to $\Gamma$:

**Theorem 6.** *A strategy $s_i \in S_i$ is root dominated if and only if it is a never best response of (at least) one $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(s_i, \check{S}_i)$ when $\epsilon \to 0^+$.*

*Proof.* The result is immediate by Lemmas 10 and 12. ■

Thus, if player $i$ believes that his opponent $j$ may have his perception of the game altered by the alternatives he considers when testing strategies, he never plays root dominated strategies.

In the different context of ordinal preferences, Börgers (1993) characterizes non rationality by weak dominance against every $j$'s strategy subset (but weak dominance is not required to be made by the same strategy). Here, in contrast, player $i$ does not restrict the game with respect to $j's$ strategies, but with respect to his own strategies (and then $j$ reacts optimally to these restrictions with probability $\epsilon$). Furthermore, it is the notion of rationality that we test against strategy subsets and not the dominance relation since the requirements of RD1′ and RD2′ are with respect to the whole game.

Besides, we can write the alternative characterization of root undominance:

**Corollary 1.** *A strategy $s_i \in S_i$ is root undominated if and only if it verifies Equation (1) when $\epsilon \to 0^+$.*

*Proof.* The result is immediate by combining Lemma 11 and Theorem 6. ■

## I.2 Deviation games

Here, we introduce our second perturbation of the game. This perturbation is such that each player believes that the opponent may observe both his "reference" strategy and the support of strategies from which a deviation might be picked by the player contemplating alternatives. In this case we will say the game is turned into a pseudo extensive form game:

**Definition 24.** *A pseudo extensive form game $\check{\Gamma}(\sigma_i^r, \check{S}_i)$ is a game where $i$ chooses a strategy in $\sigma_i^r \cup \Delta(\check{S}_i)$, where $\check{S}_i$ is a subset of $S_i$. Strategy $\sigma_i$ is the reference strategy of $i$, and $\check{S}_i$ is the support of any strategy towards which $i$ wants to deviate. Player $j$ observes this information perfectly, then forms beliefs, and plays accordingly.*

**Definition 25.** *An $\epsilon$-deviation game $\check{\Gamma}^{\epsilon}(\sigma_i^r, \check{S}_i)$ for strategy $\sigma_i^r \in \Sigma_i$ is a game where:*

1. *Player $i$ chooses the strategy $\sigma_i^r \in \Sigma_i$,*
2. *Player $i$ chooses a deviation subset $\check{S}_i \subset S_i$,*
3. *Player $i$ plays any strategy in $\sigma_i^r \cup \Delta(\check{S}_i)$,*
4. *With probability $\epsilon$, the previous steps form the first stage of a pseudo extensive form game,*
5. *Player $j$ chooses a strategy in the second stage.*

When player $j$ faces a deviation, we assume that his only concern is whether the deviation is credible according to all available information. If the deviation is *credible*, player $j$ *should* react optimally. Otherwise, he can have any belief. This last assumption does not imply that $j$ believes that $i$ has lied, or the observation is not accurate (we assume it is not possible), but rather than a non credible deviation is meaningless for $j$. In other words, it is as if $i$ said some thoughtless things that do not impact real decisions. In this case, the deviation is disregarded. Now, what do we mean by *credible*? Following Baliga and Morris (2002) and their notion of *self signaling* strategies for games with pre-play communication (see Appendix F for more details), we now introduce the notion of *self improving* strategy subset:

**Definition 26.** *A strategy subset $\check{S}_i \subset S_i$ is self improving with respect to $\sigma_i \in \Sigma_i$ if $\forall \sigma_j^* \subset \boldsymbol{b}(\check{S}_i), \exists \sigma_i''$ with $R_{\sigma_i''} = \check{S}_i$:*

$$U_i(\sigma_i'', \sigma_j^*) > U_i(\sigma_i, \sigma_j^*)$$

In words, $\check{S}_i$ is self improving with respect to $\sigma_i$ if for all best responses to $\check{S}_i$, there is a strategy whose support is $\check{S}_i$ which yields a strictly higher payoff than $\sigma_i$. Remark that if the subset $\check{S}_i$ is reduced to a singleton $\{s_i''\}$, then we have the same condition as in RD2. Furthermore, if it is the same strategy $\sigma_i''$ which strictly dominates $\sigma_i$, then we have the same condition as in RD2$'$. Since we only consider two-player games, this is always verified thanks to Pearce (1984, Lemma 3). Thus, we can equivalently write the following definition:

**Definition 27.** *A strategy $\sigma_i'' \in \Sigma_i$ is self improving with respect to $\sigma_i \in \Sigma_i$ if $\forall \sigma_j^* \subset \boldsymbol{b}(R_{\sigma_i''})$:*

$$U_i(\sigma_i'', \sigma_j^*) > U_i(\sigma_i, \sigma_j^*)$$

Instead, a strategy $\sigma_i''$ is *self signaling* when it is a best response itself (to the best response(s) played by $j$)[36]. Then, this requirement is stronger and seems to be more attractive when $j$ assesses the credibility of the deviation. However, we have to recall that the chosen strategy matters both when no deviation is observed (with probability $1-\epsilon$) and when there is deviation (with probability $\epsilon$). Therefore, it may be natural that player $j$ does not "expect" player $i$ to maximize his utility when seeing the deviation, since observation by $j$ will be made unknown for $i$. Conversely, player $j$ cannot expect that $i$ has played the deviation if whatever the optimal response he makes to this deviation, $i$'s payoff is not increased. That is why we define *credibility* in the following way:

**Definition 28.** *A deviation $\check{S}_i \in S_i$ from $\sigma_i^r \in \Sigma_i$ in a pseudo extensive form game $\check{\Gamma}(\sigma_i^r, \check{S}_i)$ is credible if there is a strategy $\sigma_i'' \in \Sigma_i$ such that:*

- $R_{\sigma_i''} = \check{S}_i$,
- *And, $\sigma_i''$ is self improving with respect to $\sigma_i^r$.*

Thus, the deviation is credible if there is a strategy $\sigma_i''$ whose support is $\check{S}_i$, and if played in the first stage of an extensive form game, pays off strictly more than $\sigma_i^r$ (given that $j$ would react optimally to $\sigma_i''$).

With the next assumption, we will restrict the beliefs of player $j$ when facing a pseudo extensive form game $\check{\Gamma}(\sigma_i, \check{S}_i)$. We assume that when the deviation is credible, the belief of $j$ that $i$ has played a strategy whose support is contained in $\check{S}_i$ is 1. In any other case, any belief is allowed. We note $\beta_i^j$ the vector which contains all the elements $\beta_i^j[\check{S}_i]$ and that represents $j$'s assessment of the probability that $i$ plays a strategy contained in $\check{S}_i$. Now, we formalize the assumption described just above:

**Assumption C.** *Player $j$, when observing a deviation $\check{S}_i \in S_i$ from $\sigma_i \in \Sigma_i$ has the following beliefs:*

- *Either $\check{S}_i$ is credible in which case $\beta_i^j[\check{S}_i | \check{S}_i \text{ is "observed"}] = 1$,*
- *Or $\check{S}_i$ is not credible in which case cells in the vector $\beta_i^j[\ |\check{S}_i \text{ is "observed"}]$ can take any value.*

Now, we define the best response of an $\epsilon$-deviation game:

[36] Of course, we make a slight abuse here because there is no notion of Best Response Set in Baliga and Morris (2002) and we take into account strategy subsets. However, since Baliga and Morris (2002) consider only pure strategies, the comparison would be relevant in their framework.

**Definition 29.** *Consider an $\epsilon$-deviation game $\check{\Gamma}^{\epsilon}(\sigma_i^r, \check{S}_i)$. A strategy $\sigma_i^* \in s_i \cup \Delta(\check{S}_i)$ is a best response of the $\epsilon$-deviation game if:*

$$\exists \sigma_j \in \Sigma_j, \exists \widehat{\sigma}_j \in \Sigma_j, \forall \sigma_i \in \Delta(\check{S}_i): \quad V_i^{\epsilon}(\sigma_i^*, \sigma_j, \widehat{\sigma}_j) \geq V_i^{\epsilon}(\sigma_i, \sigma_j, \widehat{\sigma}_j)$$

Naturally, a best response for $i$ in the $\epsilon$-deviation game is a strategy which maximizes $i$'s utility when $j$ plays $\sigma_j$ with probability $1-\epsilon$, and $\widehat{\sigma}_j$ with probability $\epsilon$. Obviously, without further restriction, any standard best response is a best response of the $\epsilon$-deviation game (think simply to cases where $\widehat{\sigma}_j = \sigma_j$). When using Assumption **C**, we can re-write the above definition in the following way:

**Lemma 13.** *Consider an $\epsilon$-deviation game $\check{\Gamma}^{\epsilon}(\sigma_i^r, \check{S}_i)$. Under Assumption* **C**, *a strategy $\sigma_i^* \in \sigma_i^r \cup \Delta(\check{S}_i)$ is a best response of the $\epsilon$-deviation game if and only if either:*

- *The deviation $\check{S}_i$ from $\sigma_i$ is credible,*
- *And,*

$$\exists \sigma_j \in \Sigma_j, \exists \sigma_j^* \subset \boldsymbol{b}(\check{S}_i), \forall \sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i): V_i^{\epsilon}(\sigma_i^*, \sigma_j, \sigma_j^*) \geq V_i^{\epsilon}(\sigma_i, \sigma_j, \sigma_j^*) \tag{cD-BR}$$

*Or,*

- *The deviation $\check{S}_i$ from $\sigma_i$ is not credible,*
- *And, $\exists \sigma_j \in \Sigma_j$ such that $\forall \sigma_i \in \sigma_i^r \cup \Delta(\check{S}_i)$: $\quad U_i(\sigma_i^*, \sigma_j) \geq U_i(\sigma_i, \sigma_j)$.*

In words, Lemma 13 means that if the deviation is credible, a best response of the $\epsilon$-deviation game $\check{\Gamma}^{\epsilon}(s_i, \check{S}_i)$ is a best response to a game where $j$ reacts optimally to $\check{S}_i$ with probability $\epsilon$. Instead, if the deviation is not credible, a best response is simply a best response according to the standard definition (see Definition 12 above) applied to $\sigma_i^r \cup \Delta(\check{S}_i)$. Remark that a best response response of a $\epsilon$-hesitation game is also a best response of the linked $\epsilon$-deviation game when the deviation is credible:

**Lemma 14.** *Consider an $\epsilon$-deviation game $\check{\Gamma}^{\epsilon}(\sigma_i^r, \check{S}_i)$. Under Assumption* **C**, *if the deviation is credible, a strategy $\sigma_i^* \in \sigma_i^r \cup \Delta(\check{S}_i)$ is a best response of the $\epsilon$-deviation game if and only if it is a best response of the $\epsilon$-hesitation game $\widehat{\Gamma}^{\epsilon}(\sigma_i^r, \check{S}_i)$.*

*Proof.* The proof is immediate since Equation (H-BR) and Equation (cD-BR) are equivalent. ∎

Besides, notice that the deviation credibility does not imply that the reference strategy $\sigma_i^r$ is a never best response of the $\epsilon$-deviation game if the deviation is credible. Now, we can show that the previous result can be applied to the reference strategy even if the deviation is not credible:

**Lemma 15.** *Under Assumption* **C**, *a strategy $\sigma_i^r \in \Sigma_i$ is a best response of an $\epsilon$-deviation game $\check{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$ if and only if it is a best response of the associated $\epsilon$-hesitation game $\widehat{\Gamma}^\epsilon(\sigma_i^r, \check{S}_i)$.*

*Proof.* First, when the deviation is credible, Lemma 14 applies. Now assume the deviation is not credible. The "if" part is straightforward. Indeed, a best response in the $\epsilon$-hesitation game is with respect to a belief with probability $1-\epsilon$ and to a belief that a best response to $\check{S}_i$ is played with probability $\epsilon$. Then, when the deviation $\check{S}_i$ from $s_i$ is not credible, any belief can be sustained, among which the one inducing that $\sigma_i^r$ is a best response of the $\epsilon$-hesitation game. Conversely, assume $\sigma_i^r$ is a best response to the considered $\epsilon$-deviation game. If the deviation is not credible, it means that there is no strategy $\sigma_i''$ whose support is $\check{S}_i$ and is self improving with respect to $\sigma_i^r$, i.e. checking $\forall \sigma_j^* \subset \boldsymbol{b}(\check{S}_i)$, $U_i(\sigma_i'', \sigma_j^*) > U_i(\sigma_i^r, \sigma_j^*)$. Thus, no strategy strictly dominates $\sigma_i^r$ when we restrict attention to $\boldsymbol{b}(\check{S}_i)$. Therefore, since it is a two-player game, by Pearce (1984, Lemma 3), $\sigma_i^r$ is a best response to at least one strategy $\sigma_j^* \in \boldsymbol{b}(\check{S}_i)$. Since $\sigma_i^r$ is also a best response to another strategy $\sigma_j$ (potentially outside $\boldsymbol{b}(\check{S}_i)$) by Lemma 13, $\sigma_i^r$ is a best response to $(\sigma_j, \sigma_j^*)$ in the $\epsilon$-hesitation game.

■

In fact, any best response of the $\epsilon$-hesitation game is also a best response of the $\epsilon$-deviation game. However, the converse is not true and the result only holds for the reference strategy $\sigma_i^r$ or when the deviation is credible.

Now, we can state the second main result of this section, still considering only two-player games:

**Theorem 7.** *Under Assumption* **C**, *a strategy $s_i \in S_i$ is root dominated if and only if it is a never best response in (at least) one $\epsilon$-deviation game $\check{\Gamma}^\epsilon(s_i, \check{S}_i)$ when $\epsilon \to 0^+$.*

*Proof.* The result is immediate by Lemma 15 and Theorem 6. ■

Theorem 7 establishes that a strategy $s_i \in S_i$ is root dominated if it is never optimal in (at least) one $0^+$-deviation game. That is, if $i$ thinks about deviations from a reference strategy and believes that these thoughts can be observed with an infinitesimal probability, he never plays root dominated strategies.

| Root Undominated $s_i \in S_i$ is equivalent to be: | |
|---|---|
| - Locally $0^+$-rational under Assumption **R** | (Theorem 5) |
| - Best Response of all $0^+$-hesitation games | (Theorem 6) |
| - Best Response of all $0^+$-deviation games under Assumption **C** | (Theorem 7) |

Figure 12: Summary of the Results

# J Rationality when $\epsilon$ moves away from 0

Now, let us examine the implications of such concepts on games outcomes when $\epsilon$ is far from 0. By contrast with the statement of Fact 1, our concepts of rationality do not refine the standard definition of rationality (see Definition 12) in this case: they are unnested. This might be seen as theoretical weakness. However, it can still be of interest in situations where experimental studies results differ from game theory predictions. The most famous example is the discrepancy between them in the prisoners' dilemma. In the dilemma, the strictly dominated strategy "cooperate" would never be rational under our concepts. Thus, the cooperation outcome would never emerge. Though, it is not because the strategy is dominated, it is because the strategy "cooperate" of *both* players is dominated. When $\epsilon$ is high enough, in the case where only one player has a strictly dominated strategy, a strictly dominated strategy can be globally rational, and the dominant strategy not globally rational as the following example shows. Global rationality may generate Pareto improvement with respect to the Nash outcome (we show in green the Nash equilibrium):

| | $j$'s Strategy | |
|---|---|---|
| $i$'s Strategy | $L$ | $R$ |
| $T$ | $(4, 1)$ | $(2, 2)$ |
| $B$ | $(3, 3)$ | $(1, 1)$ |

Table 11: Pareto efficiency of Global Rationality for intermediate values of $\epsilon$

If $\epsilon$ is high enough, but not too high, $T$ is not globally rational since the payoff of $(T, R)$ is below the payoff of $(B, L)$ (both profiles where $j$ best responds), and both $j$'s strategies are globally rational. Then, an iterated elimination of non globally rational strategies would generate the outcome $(B, L)$. However, notice that if $\epsilon$ is very high, $L$ is not globally rational anymore. That is, a consistency problem appears when players falsely firmly believe that the opponent best responds to his strategy. Additionally, it could lead to the Pareto worst outcome $(B, R)$.